Part I

Power System Analysis

CHAPTER 1

Introduction to Power System

1.1 Introduction

A power system consists of several subsystems such as generation, transmission, and distribution. The objectives of power system analysis are to model or to perform per phase analysis of power system components, to monitor the voltage at various buses, real and reactive power flow between buses, to design the circuit breakers, to plan future expansion of the existing system, to analyse the system under different fault conditions and to study the ability of the system to cope with small and large disturbances (stability studies).

1.2 Structure of Power Systems

An interconnected power system as shown in Figure 1.1 is a complex enterprise that may be subdivided into the following major subsystems:

- Generation
- Transmission and subtransmission subsystems
- Distribution subsystem
- Utilisation subsystem

Generators

An essential component of power systems is the three-phase ac generator known as synchronous generator or alternator. Its rotor is driven at synchronous speed and excited by the direct current. The other field is produced in the stator windings by the three-phase armature currents. The direct current for the rotor windings is provided by the excitation systems. In the older generators, the exciters were dc generators mounted on the same shaft, providing excitation through slip rings. The current systems use the ac generators with rotating rectifiers known as brushless excitation systems. The source of the mechanical power, commonly known as the prime mover, may be hydraulic turbines, steam turbines whose energy comes from the burning of coal, gas and nuclear fuel, gas turbines, or occasionally internal combustion engines burning oil.

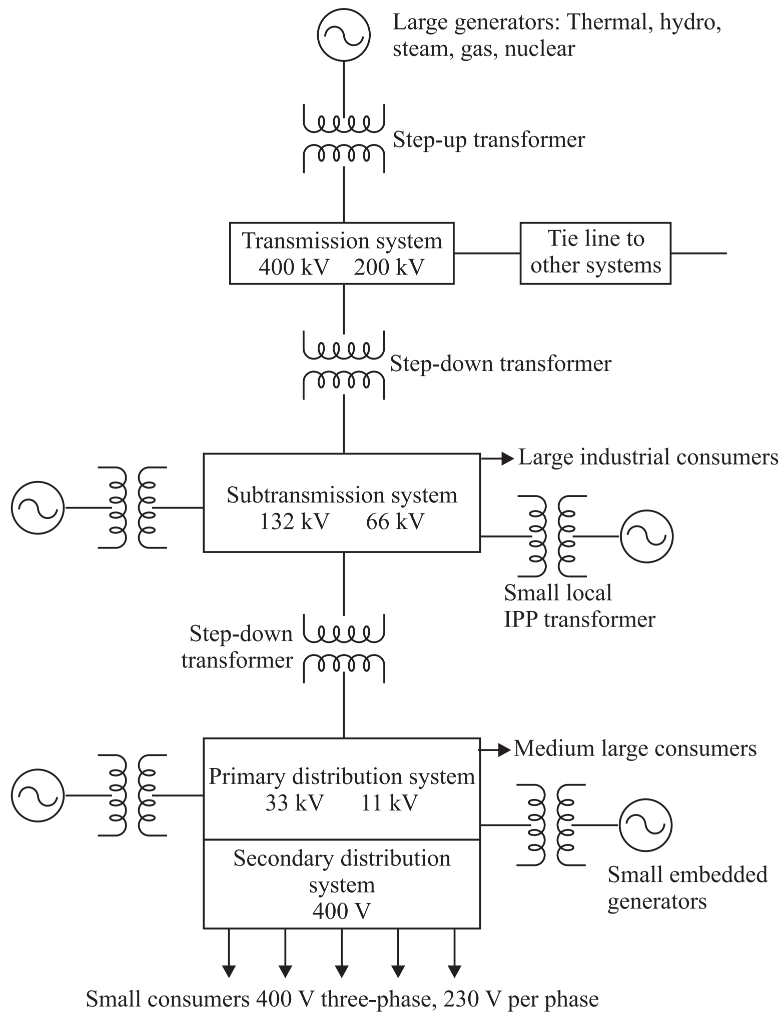


Figure 1.1 Structure of power system.

Transformers

The transformer transfers power with very high efficiency from one level of voltage to another. The power transferred to the secondary is almost the same as the primary, except for losses in the transformer. The use of a step-up transformer will reduce the losses in the transmission line, which makes the transmission of power over long distances possible.

Insulation requirements and the other practical design problems limit the generated voltage to low values, usually 11 kV. Thus, step-up transformers are used for transmission of power. At the receiving end of the transmission lines, step-down transformers are used to reduce the voltage to suitable values for distribution or utilisation. The electricity in an electric power system may undergo four or five transformations between the generator and the consumers.

Transmission and subtransmission subsystem

An overhead transmission network transfers electric power from the generating units to the distribution systems which ultimately supply the load centres at 220 kV or higher. Transmission level voltages are in the range of 66 kV to 400 kV.

As shown in Figure 1.1, electric power is generated in the range of 11 kV to 25 kV, which is increased by stepped-up transformers to the main transmission line voltage. At the substation, the connections between the various components are made, for example, lines and transformers and the arrangement for switching of these components is carried out.

The power supply network can be divided into two parts, namely, the transmission system and the distribution system. The transmission system may be further divided into primary and secondary transmission systems. The distribution system too, can be divided into primary and secondary distribution systems.

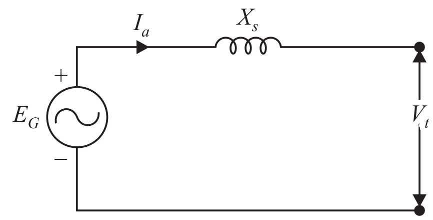
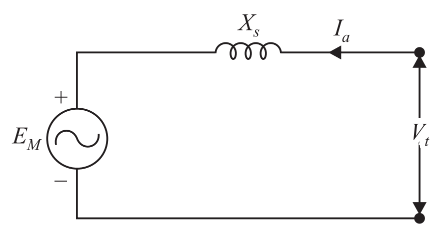
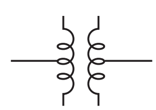
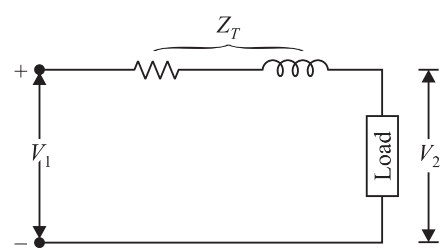
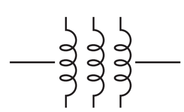
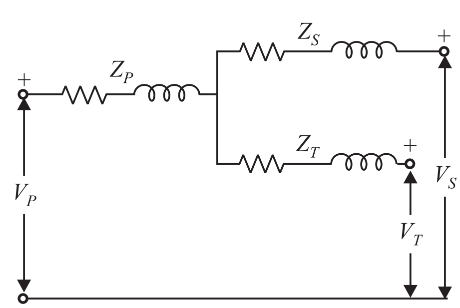
High voltage transmission lines are terminated in substations, which are called high-voltage substations, receiving substations, or primary substations. The function of some substations is switching circuits in and out of service; they are therefore referred to as switching stations. At the primary substation, the voltage is stepped down to a value more suitable for the next part of the flow towards the load. Very large industrial customers may be served directly from the primary sub-station.

The portion of the transmission system that connects the high-voltage substations through step-down transformers to the distribution substations is called the subtransmission network. Some large industrial customers may be served directly from the subtransmission system. Capacitor banks and reactor banks are usually installed in the substations for maintaining the transmission line voltage.

Distribution and utilisation subsystems

The distribution system connects the distribution substations to the consumers' service-entrance equipment. The primary distribution lines range from 3.3 to 11 kV and supply the load in a well-defined geographical area. Some small industrial customers are served directly by the primary feeders. The secondary distribution network reduces the voltage for utilisation by commercial and residential consumers. Lines and cables not exceeding a few hundred feet in length then deliver power to the individual consumers. The secondary distribution serves most of the customers at levels of 415/230 V, three phases, and four wires. The power for a typical home is derived from a transformer that reduces the primary feeder voltage to 240 V using a three-wire line. The distribution system utilises both overhead and underground conductors.

1.3 Modelling of Power System Components

S.No.	*Components*	*Single line diagram*	*Equivalent circuit or Per phase model*
1.	Synchronous generator or Alternator		
2.	Motor (synchronous motor or Induction motor)		
3.	Two winding transformer		
4.	Three winding transformer		

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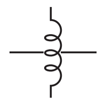
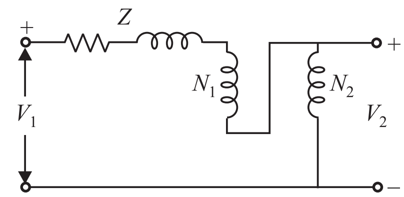
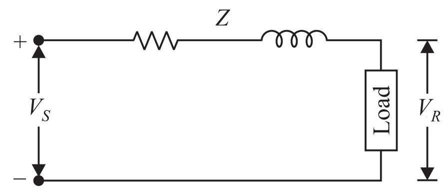
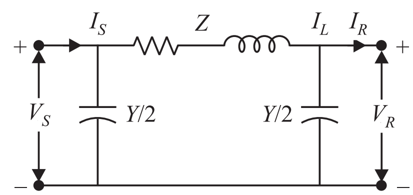
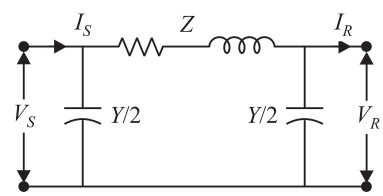
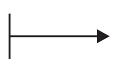
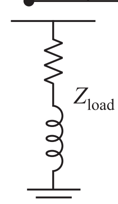
S.No.	*Components*	*Single line diagram*	*Equivalent circuit or Per phase model*
5.	Auto transformer		
6.	Short transmission line		
7.	Medium transmission line		
8.	Long transmission line		
9.	Static load		

1.4 Single Line Diagram or One Line Diagram

A single line diagram as shown in Figure 1.2 is a diagrammatic representation of a power system in which the components are represented by their symbols and the interconnections between them are shown by a straight line (even though the system might be a three-phase system). The ratings and the impedances of the components are also marked on the single line diagram. The purpose of the single line diagram is to supply the significant information about the system in a concise form.

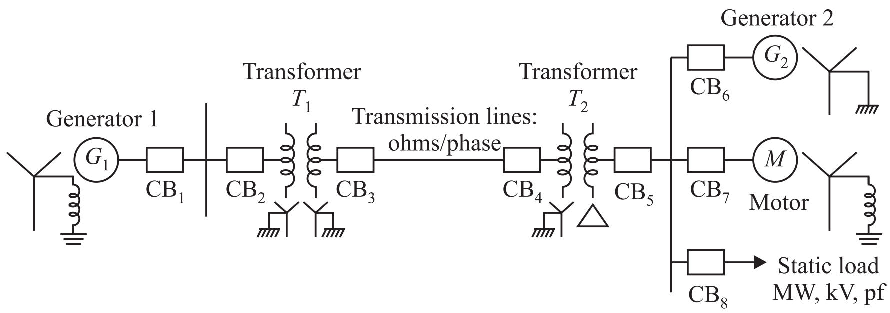


Figure 1.2 Single line representation of a representative power system.

1.4.1 Impedance Diagram

The impedance diagram (Figure 1.3) is the equivalent circuit of the power system in which the various components of the power system are represented by their approximate or simplified equivalent circuits. The impedance diagram is used for load flow studies. The following approximations are made:

(i) The neutral reactances are neglected.
(ii) The shunt branches in the equivalent circuits of transformers are neglected.

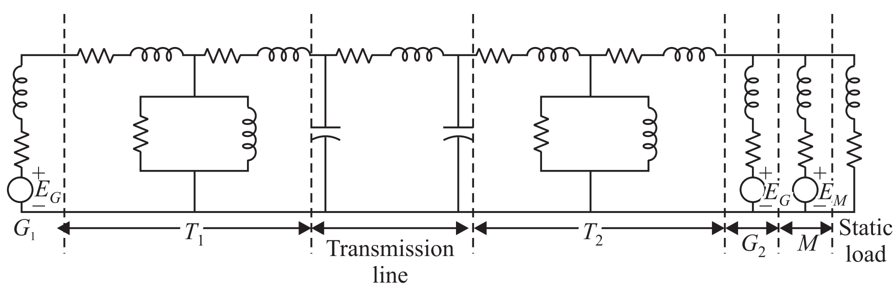


Figure 1.3 Impedance diagram of the representative power system of Figure 1.2.

1.4.2 Reactance Diagram

The reactance diagram (Figure 1.4) is the simplified equivalent circuit of the power system in which the various components of the power system are represented by their reactances. The reactance diagram can be obtained from the impedance diagram if all the resistive components are neglected. The reactance diagram is used for fault calculations. The following approximations are made:

(i) The neutral reactances are neglected.
(ii) The shunt branches in equivalent circuits of transformers are neglected.
(iii) The resistances are neglected.
(iv) All static loads are neglected.
(v) The capacitance of transmission lines is neglected.

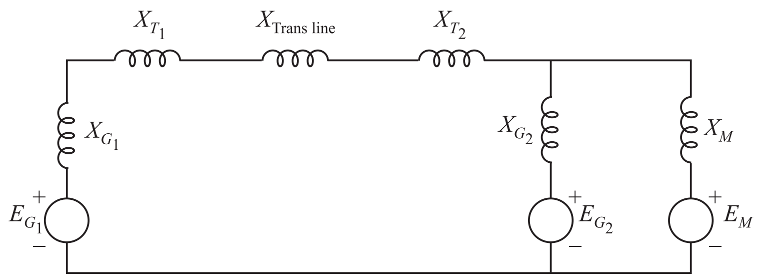


Figure 1.4 Reactance diagram of the representative power system of Figure 1.2.

1.5 Per Unit Value

The per unit value of any quantity is defined as the ratio of the actual value of that quantity to the base value of the same quantity as a decimal.

$$\text{Per unit value} = \frac{\text{Actual value}}{\text{Base value}} \tag{1.1}$$

Per phase analysis

A balanced three-phase system always analyses on per phase basis by considering one of the three-phase lines and the neutral.

Advantages of per unit system

(i) The comparison of characteristics of the various electrical apparatuses of different types of ratings is facilitated by expressing the value of reactances in per unit based on their ratings.
(ii) The per unit impedance of the transformer, whether referred to primary or secondary is the same.
(iii) The per unit system is ideal for the computerized analysis and simulation of complex power system problems.
(iv) The advantages of per unit impedance are more eagerly felt with a large number of circuits.

Single-phase system (1ϕ)

In a single phase system, suppose the base MVA and base kV ratings are given, then

$$\text{Base current (kA)} = \frac{\text{base MVA}}{\text{base kV}} \tag{1.2}$$

$$\text{Base impedance} = \frac{\text{base kV}}{\text{base kA}} = \frac{\text{base kV}}{\text{base MVA/base kV}} \tag{1.3}$$

$$\therefore \quad \text{Base impedance} = \frac{(\text{base kV})^2}{\text{base MVA}} \tag{1.4}$$

$$\text{Per unit impedance of a circuit element} = \frac{\text{actual impedance}}{\text{base impedance}}$$

$\therefore$ Impedance of a circuit element in p.u.

$$= \frac{\text{actual impedance } Z \text{ (in ohms)} \times \text{base MVA}}{(\text{base kV})^2} \tag{1.5}$$

EXAMPLE 1.1 A single-phase transformer is rated at 110/440 V, 2.5 kVA, and its leakage reactance measured from L.T. side is 0.06 Ω. Determine the leakage reactance in p.u.

Solution: Given actual leakage reactance = 0.06 Ω

$$\text{Base impedance or reactance} = \frac{(\text{base kV})^2}{\text{base MVA}} = \frac{(110 \times 10^{-3})^2}{2.5 \times 10^{-3}} = 4.84\ \Omega$$

$$\therefore \quad \text{Per unit leakage reactance} = \frac{\text{actual reactance}}{\text{base reactance}} = \frac{0.06}{4.84} = 0.0124 \text{ p.u.}$$

***Three-phase systems* (3ϕ)**

In a three-phase system, suppose the base MVA and the line-to-line base kV (L-L) ratings are given

Then, for star connection,

$$\text{Base voltage/phase} = \frac{\text{base kV (L-L)}}{\sqrt{3}} \tag{1.6}$$

$$\text{Base current/phase} = \frac{\left[\dfrac{\text{base MVA }(3\phi)}{3}\right]}{\left[\dfrac{\text{base kV (L-L)}}{\sqrt{3}}\right]} \tag{1.7}$$

$$= \frac{\text{base MVA }(3\phi)}{\sqrt{3} \times \text{base kV (L-L)}} \tag{1.8}$$

$$\text{Base impedance/phase} = \frac{\text{base voltage/phase}}{\text{base current/phase}} \tag{1.9}$$

$$= \frac{\left[\dfrac{\text{base kV (L-L)}}{\sqrt{3}}\right]}{\left[\dfrac{\text{base MVA }(3\phi)}{\sqrt{3} \times \text{base kV (L-L)}}\right]}$$

$$\therefore \quad \text{Base impedance/phase} = \frac{[\text{base kV (L-L)}]^2}{\text{base MVA }(3\phi)} \tag{1.10}$$

$$\text{Per unit impedance of a circuit element} = \frac{\text{actual impedance}}{\text{base impedance}}$$

$$\therefore \text{ Impedance in p.u.} = \frac{\text{actual impedance } Z \text{ (in ohms)} \times \text{base MVA } (3\phi)}{[\text{base kV (L-L)}]^2} \tag{1.11}$$

Change of base value

The components or various sections of power system may operate at different voltage and power levels. It will be convenient therefore for the purpose of analysis of power systems if the voltage, power, current and impedance ratings of components are expressed with reference to a common value called the base value.

$$Z_{\text{p.u.(given)}} = \frac{Z_{\text{actual}}}{(\text{base kV}_{\text{given}})^2} \times \text{base MVA}_{\text{given}} \tag{1.12}$$

Similarly, when expressed to the new base value

$$Z_{\text{p.u.(new)}} = \frac{Z_{\text{actual}}}{(\text{base kV}_{\text{new}})^2} \times \text{base MVA}_{\text{new}} \tag{1.13}$$

Dividing Eq. (1.13) by Eq. (1.12)

$$Z_{\text{p.u.(new)}} = Z_{\text{p.u.(given)}} \times \left[\frac{\text{base kV}_{\text{given}}}{\text{base kV}_{\text{new}}}\right]^2 \times \left[\frac{\text{base MVA}_{\text{new}}}{\text{base MVA}_{\text{given}}}\right] \tag{1.14}$$

For calculation of per unit values, the following points need to be noted:

1. A base kV and base MVA are selected in one part of the system. The base values for 3ϕ system are L-L kV and 3ϕ MVA.
2. The base MVA will be the same in all parts of the system.
3. For other parts of the system, i.e. on the other sides of transformers, the base kV for each part is determined using the L-L voltage ratios of the transformer.
4. The impedance values in per unit are calculated using the formulas.

EXAMPLE 1.2 Given

Generator 1: 100 MVA, 33 kV, reactance 10%
Generator 2: 150 MVA, 32 kV, reactance 8%
Generator 3: 110 MVA, 30 kV, reactance 12%

Determine the new per unit reactance of generators corresponding to the base values of 200 MVA and 35 kV.

Solution:

Base MVA, MVA_{new} = 200 MVA; Base kV, kV_{new} = 35 kV

Reactance of generator 1

$X_{\text{p.u.(given)}}$ = 10% = 0.1 p.u., $\text{MVA}_{\text{given}}$ = 100 MVA, MVA_{new} = 200 MVA, kV_{given} = 33 kV, kV_{new} = 35 kV

$$Z_{\text{p.u.(new)}} = Z_{\text{p.u.(given)}} \times \left[\frac{\text{base kV}_{\text{given}}}{\text{base kV}_{\text{new}}}\right]^2 \times \left[\frac{\text{base MVA}_{\text{new}}}{\text{base MVA}_{\text{given}}}\right]$$

$$Z_{\text{p.u.(new)}} = 0.1 \times \left[\frac{33}{35}\right]^2 \times \left[\frac{200}{100}\right] = 0.178 \text{ p.u.}$$

Reactance of generator 2

$X_{\text{p.u.(given)}}$ = 8% = 0.08 p.u., $\text{MVA}_{\text{given}}$ = 150 MVA, MVA_{new} = 200 MVA, kV_{given} = 32 kV, kV_{new} = 35 kV

$$Z_{\text{p.u.(new)}} = Z_{\text{p.u.(given)}} \times \left[\frac{\text{base kV}_{\text{given}}}{\text{base kV}_{\text{new}}}\right]^2 \times \left[\frac{\text{base MVA}_{\text{new}}}{\text{base MVA}_{\text{given}}}\right]$$

$$\therefore \quad Z_{\text{p.u.(new)}} = 0.08 \times \left[\frac{32}{35}\right]^2 \times \left[\frac{200}{150}\right] = 0.089 \text{ p.u.}$$

Reactance of generator 3

$X_{\text{p.u.(given)}}$ =12% = 0.12 p.u., $\text{MVA}_{\text{given}}$ = 110 MVA, MVA_{new} = 200 MVA, kV_{given} = 30 kV, kV_{new} = 35 kV

$$Z_{\text{p.u.(new)}} = Z_{\text{p.u.(given)}} \times \left[\frac{\text{base kV}_{\text{given}}}{\text{base kV}_{\text{new}}}\right]^2 \times \left[\frac{\text{base MVA}_{\text{new}}}{\text{base MVA}_{\text{given}}}\right]$$

$$\therefore \quad Z_{\text{p.u.(new)}} = 0.12 \times \left[\frac{30}{35}\right]^2 \times \left[\frac{200}{110}\right] = 0.16 \text{ p.u.}$$

EXAMPLE 1.3 Draw the per unit reactance diagram for the power system shown in Figure 1.5. Neglect the resistance and use a base of 100 MVA, 220 kV in a 50 Ω line. The ratings of the generator, motor and transformers are as follows:

- G : 40 MVA, 25 kV, X'' = 20%
- M : 50 MVA, 11 kV, X'' = 30%
- T_1 : 40 MVA, 33Y/220Y kV, X = 15%
- T_2 : 30 MVA, 11 Δ/220Y kV, X = 15%
- Load : 11 kV, 50 MW + j68 MVAR

Determine the new per unit values of reactance of transmission line, and new values of per unit reactance of transformer T_1, generator G, transformer T_2 and motor M.

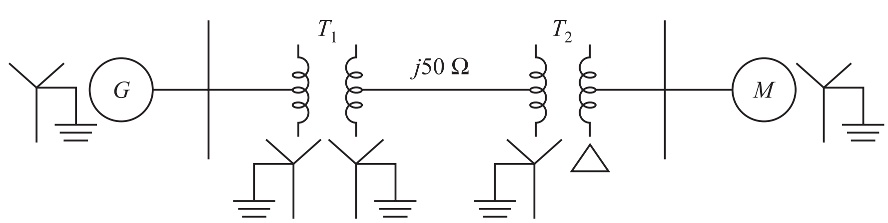


Figure 1.5 Single line diagram of Example 1.3.

Solution:

Base MVA, MVA_{new} = 100 MVA

Base kV, kV_{new} = 220 kV

Reactance of transmission line

$$\text{Per unit reactance of the transmission line} = \frac{\text{actual reactance, }\Omega}{\text{base reactance, }\Omega}$$

Actual reactance = 50 Ω

$$\text{Base reactance} = \frac{(kV_{new})^2}{MVA_{new}} = \frac{220^2}{100} = 484\ \Omega$$

$$\text{Per unit reactance of the transmission line} = \frac{\text{actual reactance, }\Omega}{\text{base reactance, }\Omega}$$

$$= \frac{50}{484} = 0.1033 \text{ p.u.}$$

Reactance of transformer T_1 (primary side)

$$X_{\text{p.u.(new)}} = X_{\text{p.u.(given)}} \times \left(\frac{kV_{given}}{kV_{new}}\right)^2 \times \left(\frac{MVA_{new}}{MVA_{given}}\right)$$

$X_{\text{p.u.(given)}}$ = 0.15 p.u., MVA_{given} = 40, MVA_{new} = 100, kV_{given} = 33
kV_{new} = ?

Base kV on LT side of transformer T_1

$$= \text{base kV on HT side} \times \frac{\text{LT voltage rating}}{\text{HT voltage rating}}$$

$$\text{Base kV on LT side of transformer } T_1 = 220 \times \frac{33}{220} = 33 \text{ kV}$$

kV_{new} = 33 kV

$$X_{\text{p.u.(new)}} = 0.15 \times \left(\frac{33}{33}\right)^2 \times \left(\frac{100}{40}\right) = 0.375 \text{ p.u.}$$

Reactance of the generator G

$$X_{\text{p.u.(new)}} = X_{\text{p.u.(given)}} \times \left(\frac{kV_{given}}{kV_{new}}\right)^2 \times \left(\frac{MVA_{new}}{MVA_{given}}\right)$$

$X_{\text{p.u.(given)}}$ = 0.2 p.u., MVA_{given} = 40, MVA_{new} = 100, kV_{given} = 25,
kV_{new} = 33

$$X_{\text{p.u.(new)}} = 0.2 \times \left(\frac{25}{33}\right)^2 \times \left(\frac{100}{40}\right) = 0.287 \text{ p.u.}$$

Reactance of transformer T_2 (primary side)

$$X_{\text{p.u.(new)}} = X_{\text{p.u.(given)}} \times \left(\frac{\text{kV}_{\text{given}}}{\text{kV}_{\text{new}}}\right)^2 \times \left(\frac{\text{MVA}_{\text{new}}}{\text{MVA}_{\text{given}}}\right)$$

$X_{\text{p.u.(given)}} = 0.15$ p.u., $\text{MVA}_{\text{given}} = 30$, $\text{MVA}_{\text{new}} = 100$, $\text{kV}_{\text{given}} = 11$, $\text{kV}_{\text{new}} = ?$

$$\text{Base kV on LT side of transformer } T_2 = \text{base kV on HT side} \times \frac{\text{LT voltages rating}}{\text{HT voltage rating}}$$

$$\text{Base kV on LT side of transformer } T_1 = 220 \times \frac{11}{220} = 11\,\text{kV}$$

$$\text{kV}_{\text{new}} = 11 \text{ kV}$$

$$X_{\text{p.u.(new)}} = 0.15 \times \left(\frac{11}{11}\right)^2 \times \left(\frac{100}{30}\right) = 0.5 \text{ p.u.}$$

Reactance of motor M

$$X_{\text{p.u.(new)}} = X_{\text{p.u.(given)}} \times \left(\frac{\text{kV}_{\text{given}}}{\text{kV}_{\text{new}}}\right)^2 \times \left(\frac{\text{MVA}_{\text{new}}}{\text{MVA}_{\text{given}}}\right)$$

$X_{\text{p.u.(given)}} = 0.3$ p.u., $\text{MVA}_{\text{given}} = 50$, $\text{MVA}_{\text{new}} = 100$, $\text{kV}_{\text{given}} = 11$, $\text{kV}_{\text{new}} = 11$

$$X_{\text{p.u.(new)}} = 0.3 \times \left(\frac{11}{11}\right)^2 \times \left(\frac{100}{50}\right) = 0.6 \text{ p.u.}$$

Reactance diagram (Figure 1.6)

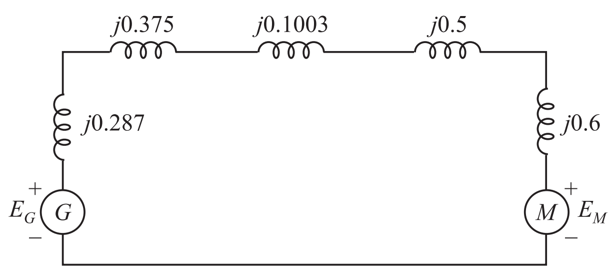


Figure 1.6 Reactance diagram of power system of Example 1.3.

EXAMPLE 1.4 Draw the reactance diagram for the power system shown in Figure 1.7. Neglect the resistance and use a base of 50 MVA and 13.8 kV on generator G_1.

G_1 : 20 MVA, 13.8 kV, $X'' = 20\%$
G_2 : 30 MVA, 18.0 kV, $X'' = 20\%$
G_3 : 30 MVA, 20.0 kV, $X'' = 20\%$

T_1 : 25 MVA, 220/13.8 kV, $X = 10\%$
T_2 : 3 single phase unit each rated 10 MVA, 127/18 kV, $X = 10\%$
T_3 : 35 MVA, 220/22 kV, $X = 10\%$

Determine the new values of per unit reactance of G_1, T_1, transmission line 1, transmission line 2, T_2, G_2, T_3 and G_3.

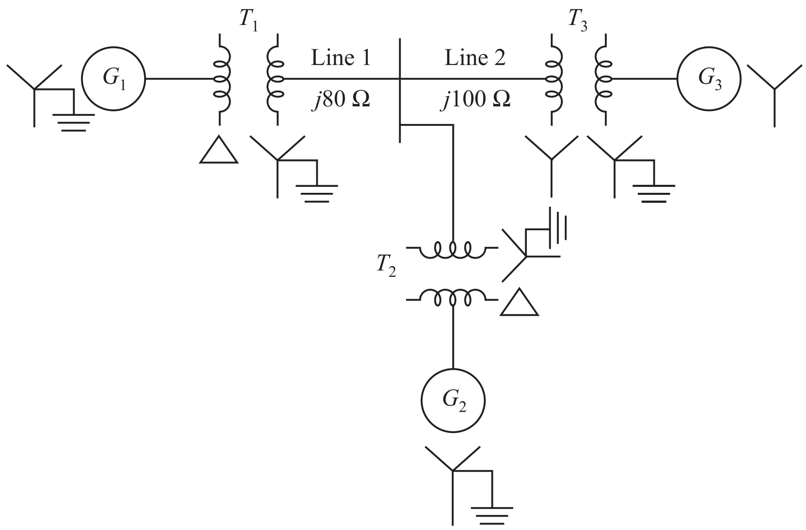


Figure 1.7 Single line diagram of Example 1.4.

Solution:

Base MVA, $\text{MVA}_{\text{new}} = 50$ MVA
Base kV, $\text{kV}_{\text{new}} = 13.8$ kV

Reactance of generator G_1

$$X_{\text{p.u.(new)}} = X_{\text{p.u.(given)}} \times \left(\frac{\text{kV}_{\text{given}}}{\text{kV}_{\text{new}}}\right)^2 \times \left(\frac{\text{MVA}_{\text{new}}}{\text{MVA}_{\text{given}}}\right)$$

$X_{\text{p.u.(given)}} = 0.2$ p.u., $\text{MVA}_{\text{given}} = 20$, $\text{MVA}_{\text{new}} = 50$, $\text{kV}_{\text{given}} = 13.8$, $\text{kV}_{\text{new}} = 13.8$

$$\therefore \qquad X_{\text{p.u.(new)}} = 0.2 \times \left(\frac{13.8}{13.8}\right)^2 \times \left(\frac{50}{20}\right) = j0.5 \text{ p.u.}$$

Reactance of transformer T_1 (primary side)

$$X_{\text{p.u.(new)}} = X_{\text{p.u.(given)}} \times \left(\frac{\text{kV}_{\text{given}}}{\text{kV}_{\text{new}}}\right)^2 \times \left(\frac{\text{MVA}_{\text{new}}}{\text{MVA}_{\text{given}}}\right)$$

$X_{\text{p.u.(given)}} = 0.1$ p.u., $\text{MVA}_{\text{given}} = 25$, $\text{MVA}_{\text{new}} = 50$, $\text{kV}_{\text{given}} = 13.8$, $\text{kV}_{\text{new}} = 13.8$

$$X_{\text{p.u.(new)}} = 0.1 \times \left(\frac{13.8}{13.8}\right)^2 \times \left(\frac{50}{25}\right) = j0.2 \text{ p.u.}$$

Reactance of the transmission line j80 Ω

$$\text{Per unit reactance of the transmission line} = \frac{\text{actual reactance, } \Omega}{\text{base reactance, } \Omega}$$

$$\text{Actual reactance} = 80\ \Omega$$

$$\text{Base kV on HT side of transformer } T_1$$

$$= \text{base kV on LT side} \times \frac{\text{HT voltage rating}}{\text{LT voltage rating}}$$

$$\text{Base kV on HT side of transformer } T_1 = 13.8 \times \frac{220}{13.8} = 220 \text{ kV}$$

$$\text{kV}_{\text{new}} = 220 \text{ kV}$$

$$\text{Base impedance} = \frac{(\text{kV}_{\text{new}})^2}{\text{MVA}_{\text{new}}} = \frac{220^2}{50} = 968\ \Omega$$

$$\text{Per unit reactance of the transmission line} = \frac{\text{actual reactance, } \Omega}{\text{base reactance, } \Omega}$$

$$= \frac{80}{968} = j0.0826 \text{ p.u.}$$

Reactance of the transmission line j100 Ω

$$\text{Per unit reactance of the transmission line} = \frac{\text{actual reactance, } \Omega}{\text{base reactance, } \Omega}$$

$$\text{Actual reactance} = 100\ \Omega$$

$$\text{Base kV on HT side of transformer } T_1$$

$$= \text{base kV on LT side} \times \frac{\text{HT voltage rating}}{\text{LT voltage rating}}$$

$$\text{Base kV on HT side of transformer } T_1 = 13.8 \times \frac{220}{13.8} = 220 \text{ kV}$$

$$\text{kV}_{\text{new}} = 220 \text{ kV}$$

$$\text{Base impedance} = \frac{(\text{kV}_{\text{new}})^2}{\text{MVA}_{\text{new}}} = \frac{220^2}{50} = 968\ \Omega$$

$$\text{Per unit reactance of the transmission line} = \frac{\text{actual reactance, } \Omega}{\text{base reactance, } \Omega}$$

$$= \frac{100}{968} = j0.1033 \text{ p.u.}$$

Reactance of transformer T_2 (primary side)

$$X_{\text{p.u.(new)}} = X_{\text{p.u.(given)}} \times \left(\frac{\text{kV}_{\text{given}}}{\text{kV}_{\text{new}}}\right)^2 \times \left(\frac{\text{MVA}_{\text{new}}}{\text{MVA}_{\text{given}}}\right)$$

$$X_{\text{p.u.(given)}} = 0.1 \text{ p.u.}$$

Y/Δ connection; voltage rating: $\sqrt{3} \times \frac{127}{18} \text{ kV} = \frac{220}{18} \text{ kV}$

$\text{MVA}_{\text{given}} = 3 \times 10 = 30$, $\text{MVA}_{\text{new}} = 50$, $\text{kV}_{\text{given}} = 220$, $\text{kV}_{\text{new}} = 220$

$$X_{\text{p.u.(new)}} = 0.1 \times \left(\frac{220}{220}\right)^2 \times \left(\frac{50}{30}\right) = j0.1667 \text{ p.u.}$$

Reactance of generator G_2

$$X_{\text{p.u.(new)}} = X_{\text{p.u.(given)}} \times \left(\frac{\text{kV}_{\text{given}}}{\text{kV}_{\text{new}}}\right)^2 \times \left(\frac{\text{MVA}_{\text{new}}}{\text{MVA}_{\text{given}}}\right)$$

$X_{\text{p.u.(given)}} = 0.2$ p.u., $\text{MVA}_{\text{given}} = 30$, $\text{MVA}_{\text{new}} = 50$, $\text{kV}_{\text{given}} = 18$, $\text{kV}_{\text{new}} = ?$

Base kV on LT side of transformer T_2

$$= \text{base kV on HT side} \times \frac{\text{LT voltage rating}}{\text{HT voltage rating}}$$

Base kV on LT side of transformer $T_2 = 220 \times \frac{18}{220} = 18 \text{ kV}$

$$\text{kV}_{\text{new}} = 18 \text{ kV}$$

$$X_{\text{p.u.(new)}} = 0.2 \times \left(\frac{18}{18}\right)^2 \times \left(\frac{50}{30}\right) = j0.333 \text{ p.u.}$$

Reactance of transformer T_3 (secondary side)

$$X_{\text{p.u.(new)}} = X_{\text{p.u.(given)}} \times \left(\frac{\text{kV}_{\text{given}}}{\text{kV}_{\text{new}}}\right)^2 \times \left(\frac{\text{MVA}_{\text{new}}}{\text{MVA}_{\text{given}}}\right)$$

$X_{\text{p.u.(given)}} = 0.1$ p.u., $\text{MVA}_{\text{given}} = 35$, $\text{MVA}_{\text{new}} = 50$, $\text{kV}_{\text{given}} = 220$, $\text{kV}_{\text{new}} = 220$

$$X_{\text{p.u.(new)}} = 0.1 \times \left(\frac{220}{220}\right)^2 \times \left(\frac{50}{35}\right) = j0.1429 \text{ p.u.}$$

Reactance of generator G_3

$$X_{\text{p.u.(new)}} = X_{\text{p.u.(given)}} \times \left(\frac{\text{kV}_{\text{given}}}{\text{kV}_{\text{new}}}\right)^2 \times \left(\frac{\text{MVA}_{\text{new}}}{\text{MVA}_{\text{given}}}\right)$$

$X_{p.u.(given)} = 0.2$ p.u., $MVA_{given} = 30$, $MVA_{new} = 50$, $kV_{given} = 20$, $kV_{new} = ?$

Base kV on LT side of transformer T_3

$$= \text{base kV on HT side} \times \frac{\text{LT voltage rating}}{\text{HT voltage rating}}$$

$$\text{Base kV on LT side of transformer } T_3 = 220 \times \frac{22}{220} = 22 \text{ kV}$$

$$kV_{new} = 22 \text{ kV}$$

$$X_{p.u.(new)} = 0.2 \times \left(\frac{20}{22}\right)^2 \times \left(\frac{50}{30}\right) = j0.2755 \text{ p.u.}$$

Reactance diagram (Figure 1.8)

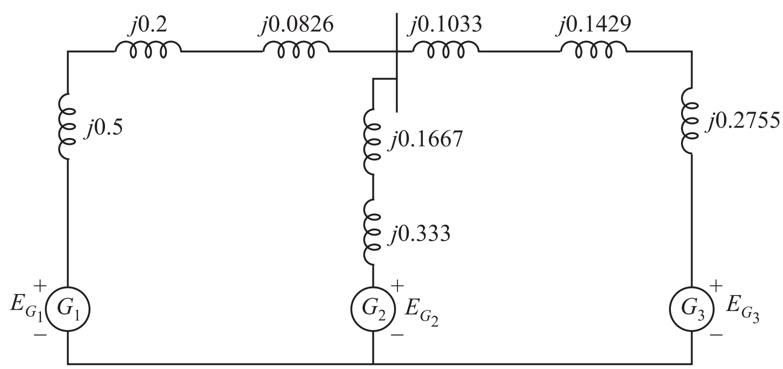


Figure 1.8 Reactance diagram of power system of Example 1.4.

EXAMPLE 1.5 A simple power system is shown in Figure 1.9. Redraw this system where the per unit reactance of the components is represented on a common 5000 VA base and common system base voltage of 250 V.

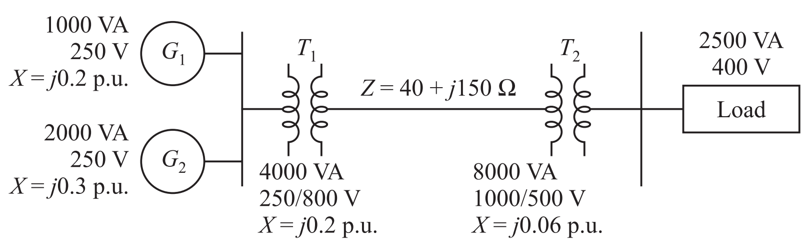


Figure 1.9 A simple power system of Example 1.5.

Determine the new values of per unit reactance of G_1, G_2, T_1, transmission line, T_2, and load.

Solution:

Base MVA, MVA_{new} = 5000 VA = 5 MVA

Base kV, kV_{new} = 250 V = 0.25 kV

Reactance of generator G_1

$$X_{\text{p.u.(new)}} = X_{\text{p.u.(given)}} \times \left(\frac{\text{kV}_{\text{given}}}{\text{kV}_{\text{new}}}\right)^2 \times \left(\frac{\text{MVA}_{\text{new}}}{\text{MVA}_{\text{given}}}\right)$$

$Z_{\text{p.u.(given)}} = 0.2$ p.u., $\text{MVA}_{\text{given}} = 1$, $\text{MVA}_{\text{new}} = 5$, $\text{kV}_{\text{given}} = 0.25$, $\text{kV}_{\text{new}} = 0.25$

$$Z_{\text{p.u.(new)}} = 0.2 \times \left(\frac{0.25}{0.25}\right)^2 \times \left(\frac{5}{1}\right) = j1.0 \text{ p.u.}$$

Reactance of generator G_2

$$X_{\text{p.u.(new)}} = X_{\text{p.u.(given)}} \times \left(\frac{\text{kV}_{\text{given}}}{\text{kV}_{\text{new}}}\right)^2 \times \left(\frac{\text{MVA}_{\text{new}}}{\text{MVA}_{\text{given}}}\right)$$

$Z_{\text{p.u.(given)}} = 0.3$ p.u., $\text{MVA}_{\text{given}} = 2$, $\text{MVA}_{\text{new}} = 5$, $\text{kV}_{\text{given}} = 0.25$, $\text{kV}_{\text{new}} = 0.25$

$$X_{\text{p.u.(new)}} = 0.3 \times \left(\frac{0.25}{0.25}\right)^2 \times \left(\frac{5}{2}\right) = j0.75 \text{ p.u.}$$

Reactance of transformer T_1 (primary side)

$$X_{\text{p.u.(new)}} = X_{\text{p.u.(given)}} \times \left(\frac{\text{kV}_{\text{given}}}{\text{kV}_{\text{new}}}\right)^2 \times \left(\frac{\text{MVA}_{\text{new}}}{\text{MVA}_{\text{given}}}\right)$$

$X_{\text{p.u.(given)}} = 0.2$ p.u., $\text{MVA}_{\text{given}} = 4$, $\text{MVA}_{\text{new}} = 5$, $\text{kV}_{\text{given}} = 0.25$, $\text{kV}_{\text{new}} = 0.25$

$$X_{\text{p.u.(new)}} = 0.2 \times \left(\frac{0.25}{0.25}\right)^2 \times \left(\frac{5}{4}\right) = j0.25 \text{ p.u.}$$

Impedance of transmission line $Z = 40 + j150\ \Omega$

$$\text{Per unit impedance of the transmission line} = \frac{\text{actual impedance, }\Omega}{\text{base impedance, }\Omega}$$

$$\text{Actual impedance} = (40 + j150)\ \Omega$$

Base V on HT side of transformer T_1

$$= \text{base V on LT side} \times \frac{\text{HT voltage rating}}{\text{LT voltage rating}}$$

$$\text{Base V on HT side of transformer } T_1 = 250 \times \frac{800}{250} = 800 \text{ V}$$

$$\text{V}_{\text{new}} = 800 \text{ V}$$

$$\text{Base impedance} = \frac{(\text{V}_{\text{new}})^2}{\text{VA}_{\text{new}}} = \frac{800^2}{5000} = 128\ \Omega$$

$$\text{Per unit impedance of the transmission line} = \frac{\text{actual impedance, }\Omega}{\text{base impedance, }\Omega}$$

$$= \frac{40 + j150}{128} = 0.3125 + j1.17 \text{ p.u.}$$

Reactance of transformer T_2 (primary side)

$$X_{\text{p.u.(new)}} = X_{\text{p.u.(given)}} \times \left(\frac{\text{kV}_{\text{given}}}{\text{kV}_{\text{new}}}\right)^2 \times \left(\frac{\text{MVA}_{\text{new}}}{\text{MVA}_{\text{given}}}\right)$$

Base kV on LT side of transformer T_2

$$= \text{Base kV on HT side} \times \frac{\text{LT voltage rating}}{\text{HT voltage rating}} = 800 \times \frac{500}{1000} = 400 = 0.4 \text{ kV}$$

$X_{\text{p.u.(given)}} = 0.06$ p.u., $\text{MVA}_{\text{given}} = 8$, $\text{MVA}_{\text{new}} = 5$, $\text{kV}_{\text{given}} = 0.5$, $\text{kV}_{\text{new}} = 0.4$

$$X_{\text{p.u.(new)}} = 0.06 \times \left(\frac{0.5}{0.4}\right)^2 \times \left(\frac{5}{8}\right) = j0.0585 \text{ p.u.}$$

Reactance diagram (Figure 1.10)

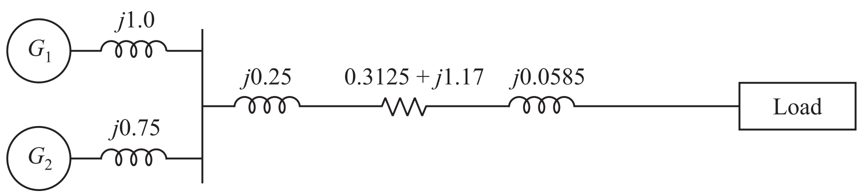


Figure 1.10 Reactance diagram of power system of Example 1.5.

EXAMPLE 1.6 The single line diagram of a three-phase power system is shown in Figure 1.11. Select a common base of 100 MVA and 13.8 kV on the generator side. Draw the per unit impedance diagram with new values per unit reactances.

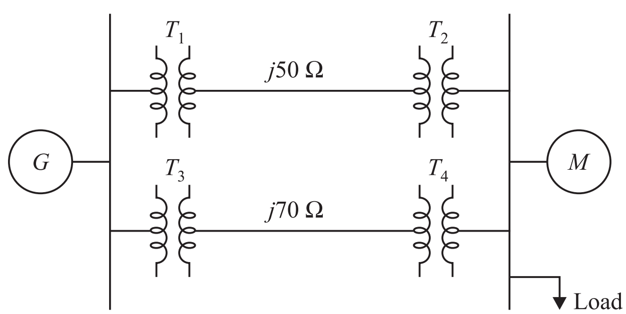


Figure 1.11 Single line diagram of power system of Example 1.6.

G : 90 MVA, 13.8 kV, $X = 18\%$

T_1 : 50 MVA, 13.8/220 kV, $X = 10\%$

T_2 : 50 MVA, 220/11 kV, $X = 10\%$

T_3 : 50 MVA, 13.8/132 kV, X = 10%
T_4 : 50 MVA, 132/11 kV, X = 10%
M : 80 MVA, 10.45 kV, X = 20%
Load : 57 MVA, 0.8 p.f lagging at 10.45 kV
Line 1=j50 Ω; Line 2 = j70 Ω

Solution:

Base MVA, MVA_{new} = 100 MVA
Base kV, kV_{new} = 13.8 kV

Reactance of generator G_1

$$X_{p.u.(new)} = X_{p.u.(given)} \times \left(\frac{kV_{given}}{kV_{new}}\right)^2 \times \left(\frac{MVA_{new}}{MVA_{given}}\right)$$

$X_{p.u.(given)}$ = 0.18 p.u., MVA_{given} = 90, MVA_{new} = 100, kV_{given} = 13.8, kV_{new} = 13.8

$$X_{p.u.(new)} = 0.18 \times \left(\frac{13.8}{13.8}\right)^2 \times \left(\frac{100}{90}\right) = j0.2 \text{ p.u.}$$

Reactance of transformer T_1 (primary side)

$$X_{p.u.(new)} = X_{p.u.(given)} \times \left(\frac{kV_{given}}{kV_{new}}\right)^2 \times \left(\frac{MVA_{new}}{MVA_{given}}\right)$$

$X_{p.u.(given)}$ = 0.1 p.u., MVA_{given} = 50, MVA_{new} = 100, kV_{given} = 13.8, kV_{new} = 13.8

$$X_{p.u.(new)} = 0.1 \times \left(\frac{13.8}{13.8}\right)^2 \times \left(\frac{100}{50}\right) = j0.2 \text{ p.u.}$$

Reactance of transmission line j50 Ω

$$\text{Per unit reactance of the transmission line} = \frac{\text{actual reactance, }\Omega}{\text{base reactance, }\Omega}$$

Actual reactance = 50 Ω

Base kV on HT side of transformer T_1

$$= \text{base kV on LT side} \times \frac{\text{HT voltage rating}}{\text{LT voltage rating}}$$

$$\text{Base kV on HT side of transformer } T_1 = 13.8 \times \frac{220}{13.8} = 220 \text{ kV}$$

kV_{new} = 220 kV

$$\text{Base impedance} = \frac{(kV_{new})^2}{MVA_{new}} = \frac{220^2}{100} = 484\ \Omega$$

$$\text{Per unit reactance of the transmission line} = \frac{\text{actual reactance, }\Omega}{\text{base reactance, }\Omega}$$

$$= \frac{50}{484} = j0.1033 \text{ p.u.}$$

Reactance of transformer T_2 (secondary side)

$$X_{\text{p.u.(new)}} = X_{\text{p.u.(given)}} \times \left(\frac{\text{kV}_{\text{given}}}{\text{kV}_{\text{new}}}\right)^2 \times \left(\frac{\text{MVA}_{\text{new}}}{\text{MVA}_{\text{given}}}\right)$$

$X_{\text{p.u.(given)}} = 0.1$ p.u., $\text{MVA}_{\text{given}} = 50$, $\text{MVA}_{\text{new}} = 100$,
$\text{kV}_{\text{given}} = 220$,
$\text{kV}_{\text{new}} = 220$

$$X_{\text{p.u.(new)}} = 0.1 \times \left(\frac{220}{220}\right)^2 \times \left(\frac{100}{50}\right) = j0.2 \text{ p.u.}$$

Reactance of transformer T_3 (primary side)

$$X_{\text{p.u.(new)}} = X_{\text{p.u.(given)}} \times \left(\frac{\text{kV}_{\text{given}}}{\text{kV}_{\text{new}}}\right)^2 \times \left(\frac{\text{MVA}_{\text{new}}}{\text{MVA}_{\text{given}}}\right)$$

$X_{\text{p.u.(given)}} = 0.1$ p.u., $\text{MVA}_{\text{given}} = 50$, $\text{MVA}_{\text{new}} = 100$, $\text{kV}_{\text{given}} = 13.8$,
$\text{kV}_{\text{new}} = 13.8$

$$X_{\text{p.u.(new)}} = 0.1 \times \left(\frac{13.8}{13.8}\right)^2 \times \left(\frac{100}{50}\right) = j0.2 \text{ p.u.}$$

Reactance of transmission line j70 Ω

$$\text{Per unit reactance of the transmission line} = \frac{\text{actual reactance, }\Omega}{\text{base reactance, }\Omega}$$

Actual reactance = 70 Ω

Base kV on HT side of transformer T_3

$$= \text{base kV on LT side} \times \frac{\text{HT voltage rating}}{\text{LT voltage rating}}$$

$$\text{Base kV on HT side of transformer } T_3 = 13.8 \times \frac{132}{13.8} = 132 \text{ kV}$$

$\text{kV}_{\text{new}} = 132$ kV

$$\text{Base impedance} = \frac{(\text{kV}_{\text{new}})^2}{\text{MVA}_{\text{new}}} = \frac{132^2}{100} = 174.24\ \Omega$$

$$\text{Per unit reactance of the transmission line} = \frac{\text{actual reactance, }\Omega}{\text{base reactance, }\Omega}$$

$$= \frac{70}{174.24} = j0.4017 \text{ p.u.}$$

Reactance of transformer T_4 (secondary side)

$$X_{\text{p.u.(new)}} = X_{\text{p.u.(given)}} \times \left(\frac{\text{kV}_{\text{given}}}{\text{kV}_{\text{new}}}\right)^2 \times \left(\frac{\text{MVA}_{\text{new}}}{\text{MVA}_{\text{given}}}\right)$$

$X_{\text{p.u.(given)}} = 0.1$ p.u., $\text{MVA}_{\text{given}} = 50$, $\text{MVA}_{\text{new}} = 100$, $\text{kV}_{\text{given}} = 132$;
$\text{kV}_{\text{new}} = 132$

$$X_{\text{p.u.(new)}} = 0.1 \times \left(\frac{132}{132}\right)^2 \times \left(\frac{100}{50}\right) = j0.2 \text{ p.u.}$$

Reactance of motor M

$$X_{\text{p.u.(new)}} = X_{\text{p.u.(given)}} \times \left(\frac{\text{kV}_{\text{given}}}{\text{kV}_{\text{new}}}\right)^2 \times \left(\frac{\text{MVA}_{\text{new}}}{\text{MVA}_{\text{given}}}\right)$$

$X_{\text{p.u.(given)}} = 0.2$ p.u., $\text{MVA}_{\text{given}} = 80$, $\text{MVA}_{\text{new}} = 100$,
$\text{kV}_{\text{given}} = 10.45$
$\text{kV}_{\text{new}} = ?$

Base kV on LT side of transformer T_4

$$= \text{base kV on HT side} \times \frac{\text{LT voltage rating}}{\text{HT voltage rating}}$$

$$\text{Base kV on LT side of transformer } T_4 = 132 \times \frac{11}{132} = 11 \text{ kV}$$

$$\text{kV}_{\text{new}} = 11 \text{ kV}$$

$$X_{\text{p.u.(new)}} = 0.2 \times \left(\frac{10.45}{11}\right)^2 \times \left(\frac{100}{80}\right) = j0.2256 \text{ p.u.}$$

The load at 0.8 p.f lagging is given by

$$S_L(3\phi) = 57\angle 36.87°$$

Load impedance is given by

$$Z_L = \frac{(V_{L-L})^2}{S_{L(3\phi)}} = \frac{10.45^2}{57\angle 36.87°} = (1.532 + j1.1495)\Omega$$

Base impedance for the load is given by

$$\text{Base impedance} = \frac{(\text{kV}_{\text{new}})^2}{\text{MVA}_{\text{new}}} = \frac{11^2}{100} = 1.21\,\Omega$$

$$\text{Per unit reactance of the transmission line} = \frac{\text{actual reactance, }\Omega}{\text{base reactance, }\Omega}$$

$$= \frac{1.532 + j1.1495}{1.21}$$

$$= (1.266 + j0.95) \text{ p.u.}$$

Reactance diagram (Figure 1.12)

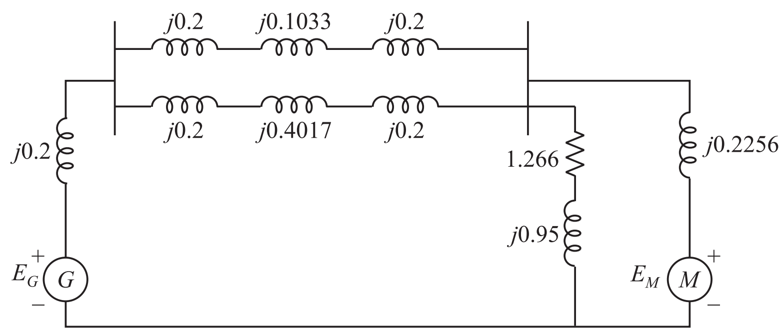


Figure 1.12 Reactance diagram of power system of Example 1.6.

1.6 Network Modelling

1.6.1 Bus Frame of Analysis

For a network with n number of nodes (buses) excluding the reference node, a set of following equations, one for each node can be written as

$$\begin{aligned} I_1 &= Y_{11}V_1 + Y_{12}V_2 + \cdots + Y_{1n}V_n \\ I_2 &= Y_{21}V_1 + Y_{22}V_2 + \cdots + Y_{2n}V_n \\ &\;\;\vdots \qquad\quad \vdots \qquad\qquad\quad \vdots \\ I_n &= Y_{n1}V_1 + Y_{n2}V_2 + \cdots + Y_{nn}V_n \end{aligned} \tag{1.15}$$

i.e. $$I_i = \sum_{m=1}^{n} Y_{im}V_m \qquad i = 1, 2, 3 \ldots, n \tag{1.16}$$

where I_i is the current entering the ith bus
V_m is the voltage to reference of bus m
Y_{im} is the admittance between the buses i and m.

In matrix form, $$\begin{bmatrix} I_1 \\ I_2 \\ \vdots \\ I_n \end{bmatrix} = \begin{bmatrix} Y_{11} & Y_{12} & \cdots & Y_{1n} \\ Y_{21} & Y_{22} & \cdots & Y_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ Y_{n1} & Y_{n2} & \cdots & Y_{nn} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ \vdots \\ V_n \end{bmatrix} \tag{1.17}$$

i.e. $$I_{\text{bus}} = Y_{\text{bus}}\, V_{\text{bus}} \tag{1.18}$$

where Y_{bus} is the bus admittance matrix.

i.e. $$Y_{\text{bus}} = \begin{bmatrix} Y_{11} & Y_{12} & \cdots & Y_{1n} \\ Y_{21} & Y_{22} & \cdots & Y_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ Y_{n1} & Y_{n2} & \cdots & Y_{nn} \end{bmatrix} \tag{1.19}$$

Now, $$V_{\text{bus}} = Z_{\text{bus}}\, I_{\text{bus}} \tag{1.20}$$

where Z_{bus} is the bus impedance matrix.

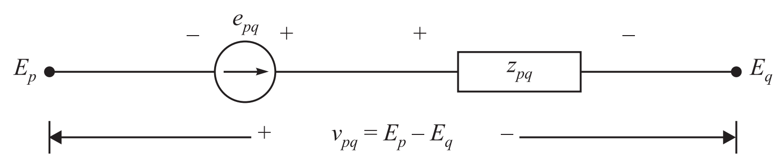
The diagonal elements of bus admittance matrix Y_{11}, Y_{22}, ..., Y_{nn} are called the short-circuit driving point admittances of the system bases, and the off-diagonal elements are called the short-circuit transfer admittances. Similarly, the diagonal elements of bus impedance matrices Z_{11}, Z_{22}, ..., Z_{nn} are called open circuit driving point impedances of the system bases, and the off-diagonal elements are known as open circuit transfer impedances.

To find out the elements of Z_{bus} and Y_{bus}, we need

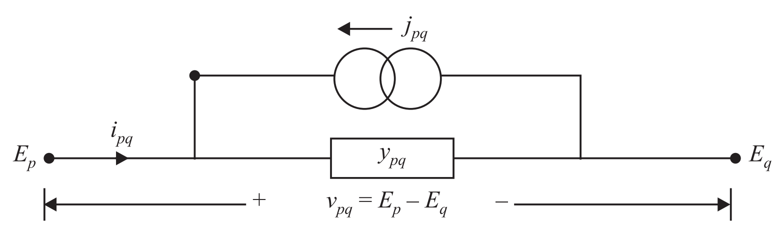
1. Primitive network
2. Graph theory
3. Incidence matrices

1.6.2 Primitive Network

A network element may in general contain active and passive components. Network components are represented both in impedance form and in admittance form as shown in Figure 1.13.



(a) Impedance form



(b) Admittance form

Figure 1.13 Primitive network.

In Figure 1.13, we have used the following notations:

v_{pq} = voltage across the element p-q
e_{pq} = voltage source in series with the element p-q
i_{pq} = current through the element p-q
z_{pq} = self impedance of the element p-q
j_{pq} = current source in parallel with the element p-q
y_{pq} = self admittance of the element p-q

In steady state condition, the variables v_{pq} and i_{pq} and the parameters of the elements z_{pq} and y_{pq} are real numbers for dc and complex numbers for ac.

The performance equation of the element in impedance form is

$$v_{pq} + e_{pq} = z_{pq} \times i_{pq} \tag{1.21}$$

The performance equation of the element in admittance form is

$$i_{pq} + j_{pq} = y_{pq} \times v_{pq} \tag{1.22}$$

The parallel current source in admittance form is related to the series voltage source in impedance form as

$$i_{pq} + j_{pq} = y_{pq} \times v_{pq}$$

or
$$j_{pq} = -i_{pq} + (y_{pq} \times v_{pq})$$

or
$$j_{pq} = -\left(\frac{v_{pq} + e_{pq}}{z_{pq}}\right) + (y_{pq} \times v_{pq})$$

$\therefore$
$$j_{pq} = -\, y_{pq} \times e_{pq} \tag{1.23}$$

A set of unconnected elements is defined as a primitive network. The performance equations of a primitive network can be derived from the above equations by expressing the variables as vectors and parameters as matrices.

The performance in impedance form $\mathbf{v} + \mathbf{e} = [z]\,\mathbf{i}$ (1.24)

The performance in admittance form $\mathbf{i} + \mathbf{j} = [y]\,\mathbf{v}$ (1.25)

Here $[z]$ and $[y]$ are primitive impedance and primitive admittance matrices, respectively, of the network.

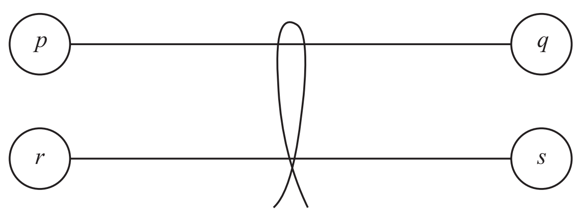


Figure 1.14 Mutual coupling between the elements *pq* and *rs*.

The mutual coupling between elements *pq* and *rs* is shown in Figure 1.14. The diagonal element of the matrix $[z]$ or matrix $[y]$ of the primitive network is the self impedance $z_{pq,\,pq}$ or self admittance $y_{pq,\,pq}$. The off-diagonal element is the mutual impedance $z_{pq,\,rs}$ or mutual admittance $y_{pq,\,rs}$ between the elements *pq* and *rs*.

1.6.3 Network Graph Theory

The geometrical structure of a network is sufficient to replace the network components by a single line segment irrespective of the characteristic of the components. These line segments are called ***element*** and their terminals are called ***nodes***. A node and an element are ***incident*** if the node is the terminal of the element. Nodes can be incident to one or more elements.

A ***graph*** shows the geometrical interconnection of the element of a network. The *rank of a graph* is $n - 1$, where n is the number of nodes in the graph. A ***subgraph*** is any subset of the graph. If each element of the connected graph is assigned a direction, it is then called ***oriented graph***. A graph is said to be ***planar***, if it can be drawn without crossover of edges, otherwise it is called non-planar.

Figure 1.15(a) shows the single line diagram of a simple power network consisting of generating stations, transformer, transmission lines and loads. Figure 1.15(b) shows the positive sequence network of the system depicted in Figure 1.15(a). The oriented connected graph is shown in Figure 1.16 for the same system.

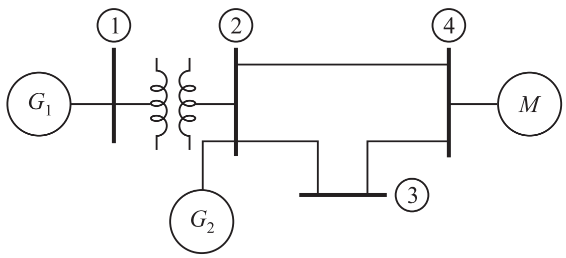


Figure 1.15(a) Sample single line diagram.

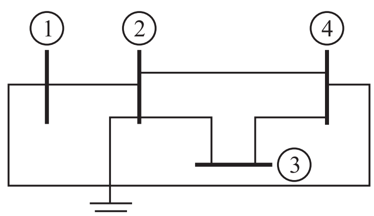


Figure 1.15(b) Positive sequence network diagram.

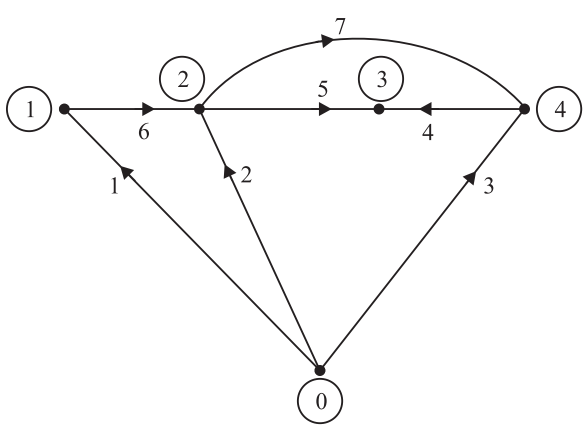


Figure 1.16 Oriented graph.

Tree and co-tree

A tree is a connected subgraph of a network which consists of all nodes of the original graph but no closed path. The graph of a network may have a number of trees. In general, if a tree contains n nodes, then it has $(n - 1)$ branches.

In forming a tree for a given graph, certain branches are removed. The branches thus opened are called links or link branches. The link for Figure 1.17, for example, is 5, 6 and 7. The set of all links of a given tree is called the co-tree of the graph.

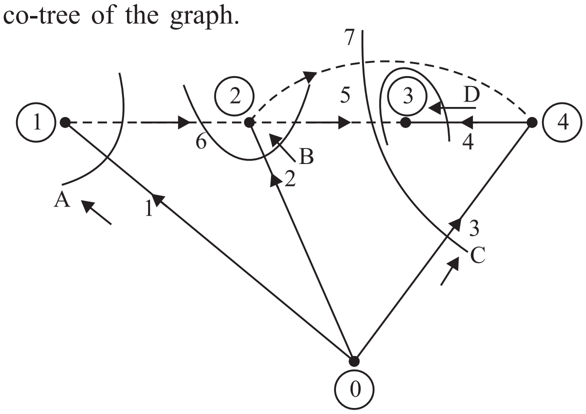


Figure 1.17 Tree of the representative power system of Figure 1.15(a).

The relation between the number of nodes and the number of branches in a tree is given by

$$b = n - 1 \tag{1.26}$$

If e is the total number of elements, then the number of links l of a connected graph with branches b is given by

$$l = e - b \tag{1.27}$$

Hence, from Eq. (1.26), the number of links l can be written as

$$l = e - n + 1 \tag{1.28}$$

A tree and the corresponding co-tree of the graph for the system are shown in Figure 1.17 and Figure 1.18.

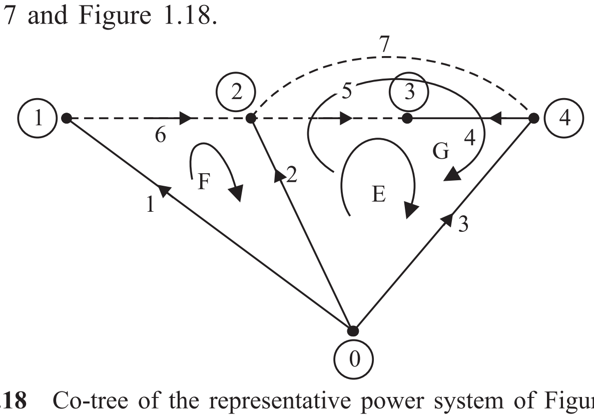


Figure 1.18 Co-tree of the representative power system of Figure 1.15(a).

1.6.4 Incidence Matrices

Incidence matrices are mostly used in graph theory. They are significant in developing the different networks matrices such as bus admittance matrix, bus impedance matrix using singular or non-singular transformation.

The following incidence matrices are of interest in power system analysis.

1. Element–node incidence matrices
2. Bus incidence matrix or element–bus incidence matrix
3. Basic loop incidence matrix
4. Basic cut-set incidence matrix

Element–node incidence matrices ($\bar{A}$)

The incidence of elements to nodes in a connected graph is shown in Figure 1.15 by the element–node incidence matrix. The elements of the matrix are as follows:

$a_{ij} = 1$ if the ith element is incident to and oriented away from the jth node

$a_{ij} = -1$ if the ith element is incident to and oriented towards the jth node

$a_{ij} = 0$ if the ith element is not incident to the jth node

The dimension of the matrix $\bar{A}$ is $e \times n$, where $\boldsymbol{e}$ is the number of elements and n is the number of nodes in the graph.

$$\bar{A} = \begin{array}{c|ccccc} e \backslash n & 0 & 1 & 2 & 3 & 4 \\ \hline 1 & 1 & -1 & 0 & 0 & 0 \\ 2 & 1 & 0 & -1 & 0 & 0 \\ 3 & 1 & 0 & 0 & 0 & -1 \\ 4 & 0 & 0 & 0 & -1 & 1 \\ 5 & 0 & 0 & 1 & -1 & 0 \\ 6 & 0 & 1 & -1 & 0 & 0 \\ 7 & 0 & 0 & 1 & 0 & -1 \end{array}$$

Since

$$\sum_{j=0}^{4} a_{ij} = 0 \qquad i = 1, 2, 3, \dots, n$$

The columns of $\bar{A}$ are linearly dependent. Hence, the rank of $\bar{A} < n$.

Bus incidence matrix or element–bus incidence matrix (**A**)

Any node of a connected graph can be selected as the reference node. Then, the variables of the other nodes are referred to as buses. The matrix obtained from the element node incident matrix ($\bar{A}$) by deleting the columns corresponding to the reference node is the element bus incidence matrix or bus incidence matrix.

The dimension of the matrix A is $e \times (n - 1)$ or $e \times b$. Node **0** is the reference node.

$$A = \begin{array}{c|cccc} e \backslash \text{Bus} & 1 & 2 & 3 & 4 \\ \hline 1 & -1 & 0 & 0 & 0 \\ 2 & 0 & -1 & 0 & 0 \\ 3 & 0 & 0 & 0 & -1 \\ 4 & 0 & 0 & -1 & 1 \\ 5 & 0 & 1 & -1 & 0 \\ 6 & 1 & -1 & 0 & 0 \\ 7 & 0 & 1 & 0 & -1 \end{array}$$

Basic loop incidence matrix (C)

The incidence of elements to basic loops of a connected graph is shown in Figure 1.19 by the basic loop incidence matrix. The elements of this matrix are:

$C_{ij} = 1$ if the ith element is incident to and oriented in the same direction as the jth basic loop

$C_{ij} = -1$ if the ith element is incident to and oriented in the opposite direction as the jth basic loop

$C_{ij} = 0$ if the ith element is not incident to the jth basic loop

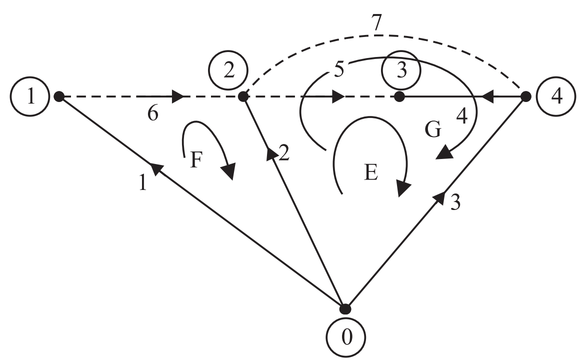


Figure 1.19 Basic loops E, F and G.

$$C = \begin{array}{c|ccc} \text{Element} \backslash \text{Basic loop} & E & F & G \\ \hline 1 & 0 & 1 & 0 \\ 2 & 1 & -1 & 1 \\ 3 & -1 & 0 & -1 \\ 4 & -1 & 0 & 0 \\ 5 & 1 & 0 & 0 \\ 6 & 0 & 1 & 0 \\ 7 & 0 & 0 & 1 \end{array}$$

Basic cut-set incidence matrix (D)

The incidence of elements to basic cut-set of a connected graph is shown in Figure 1.20 by the basic cut-set incidence matrix. The elements of this matrix are:

$d_{ij} = 1$ if the ith element is incident to the jth basic cut-set and oriented in the same direction of the jth basic cut-set

$d_{ij} = 1$ if the ith element is incident to the jth basic cut-set and oriented in the opposite direction of the jth basic cut-set

$d_{ij} = 0$ if the ith element is not incident to the jth basic cut-set

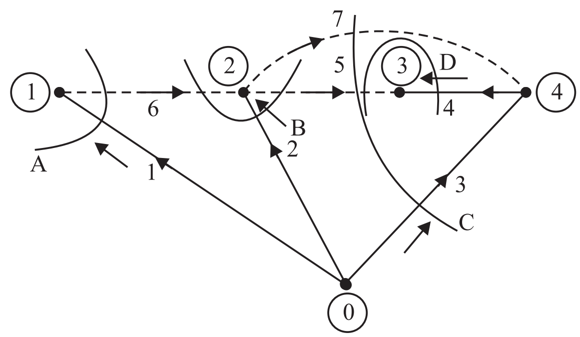


Figure 1.20 Basic cut-set of connected graph.

	Element \ Basic cut-set	A	B	C	D
	1	1	0	0	0
	2	0	1	0	0
	3	0	0	1	0
D =	4	0	0	0	1
	5	0	–1	1	1
	6	–1	1	0	0
	7	0	0	1	0

1.7 Formation of Bus Admittance Matrix [Y_{bus}]

The matrix consisting of the self admittance and the mutual admittance of the network of the power system is called the bus admittance matrix Y_{bus}. We will discuss here the following two methods by formulation of [Y_{bus}].

- Direct inspection method
- Singular transformation method (Primitive network)

1.7.1 Direct Inspection Method

Consider a simple three bus system (Figure 1.21).

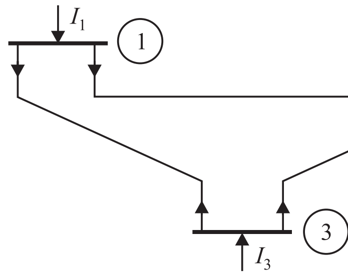


Figure 1.21 Simple three bus system.

Let I_1, I_2 and I_3 denote the current flowing into the buses. Applying KCL at each node:

At node 1:

$$I_1 = I_{11} + I_{12} + I_{13} \tag{1.29}$$
$$= y_{11}V_1 + (V_1 - V_2)y_{12} + (V_1 - V_3)y_{13}$$
$$= y_{11}V_1 + y_{12}V_1 - y_{12}V_2 + y_{13}V_1 - y_{13}V_3$$
$$= V_1(y_{11} + y_{12} + y_{13}) - y_{12}V_2 - y_{13}V_3$$

$$\therefore \quad I_1 = V_1Y_{11} + V_2Y_{12} + V_3Y_{13} \tag{1.30}$$

where

$$Y_{11} = y_{11} + y_{12} + y_{13} \quad \text{(Shunt charging admittance at bus)} \tag{1.31}$$
$$Y_{12} = -y_{12};\ Y_{13} = -y_{13} \tag{1.32}$$

At node 2:

$$I_2 = I_{22} + I_{21} + I_{23} \tag{1.33}$$
$$= y_{22}V_2 + (V_2 - V_1)y_{21} + (V_2 - V_3)y_{23}$$
$$= y_{22}V_2 + y_{21}V_2 - y_{21}V_1 + y_{23}V_2 - y_{23}V_3$$
$$= -y_{21}V_1 + V_2(y_{22} + y_{21} + y_{23}) - y_{23}V_3$$

$$\therefore \quad I_2 = V_1Y_{21} + V_2Y_{22} + V_3Y_{23} \tag{1.34}$$

where

$$Y_{22} = y_{21} + y_{22} + y_{23} \quad \text{(Shunt charging admittance at bus)} \tag{1.35}$$
$$Y_{21} = -y_{21};\ Y_{23} = -y_{23} \tag{1.36}$$

At node 3:

$$I_3 = I_{33} + I_{31} + I_{32} \tag{1.37}$$
$$= y_{33}V_3 + (V_3 - V_1)y_{31} + (V_3 - V_2)y_{32}$$
$$= y_{33}V_3 + y_{31}V_3 - y_{31}V_1 + y_{32}V_3 - y_{32}V_2$$
$$= -y_{31}V_1 - y_{32}V_2 + V_3(y_{33} + y_{31} + y_{32})$$

$$\therefore \quad I_3 = V_1Y_{31} + V_2Y_{32} + V_3Y_{33} \tag{1.38}$$

where

$$Y_{33} = y_{31} + y_{32} + y_{33} \quad \text{(Shunt charging admittance at bus)} \tag{1.39}$$
$$Y_{31} = -y_{31};\ Y_{32} = -y_{32} \tag{1.40}$$

The obtained nodal equations are now represented in matrix form as

$$\begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} = \begin{bmatrix} Y_{11} & Y_{12} & Y_{13} \\ Y_{21} & Y_{22} & Y_{23} \\ Y_{31} & Y_{32} & Y_{33} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} \tag{1.41}$$

i.e.
$$[I_{bus}] = [Y_{bus}]\,[V_{bus}] \tag{1.42}$$

or
$$I_i = \sum_{j=1}^{n} Y_{ij} V_j \qquad i = 1, 2, 3, ..., n \tag{1.43}$$

Self admittance: The terms Y_{ii} are self admittance of respective nodes and represent the algebraic sum of all the admittances connected to that node. Each diagonal term in the $[Y_{bus}]$ matrix is the self admittance term.

$$Y_{ii} = \sum_{j=1}^{n} y_{ij} \tag{1.44}$$

Mutual admittance: The mutual admittance between two buses is the negative of the sum of all the admittances connected directly between these two buses. All the non-diagonal terms in the $[Y_{bus}]$ matrix are the mutual admittance terms.

$$Y_{ij} = -y_{ij} \tag{1.45}$$

EXAMPLE 1.7 Determine the bus admittance matrix $[Y_{bus}]$ of the representative power system shown in Figure 1.22. Data for this system is given in Table 1.1.

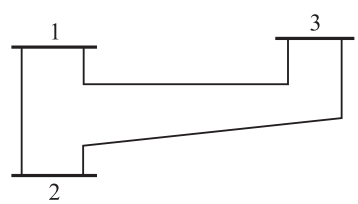


Figure 1.22 Representative power system: Example 1.7.

Table 1.1 Data for Example 1.7

Bus code i–k	***Impedance, Z_{ik}***	***Line charging $y_{ij}/2$***
1–2	$0.02 + j0.06$	$j0.03$
1–3	$0.08 + j0.24$	$j0.025$
2–3	$0.06 + j0.18$	$j0.020$

Solution:

$$y_{11} = \frac{y'_{12}}{2} + \frac{y'_{13}}{2} = j0.03 + j0.025 = j0.055$$

$$y_{22} = \frac{y'_{21}}{2} + \frac{y'_{23}}{2} = j0.03 + j0.020 = j0.050$$

$$y_{33} = \frac{y'_{31}}{2} + \frac{y'_{32}}{2} = j0.025 + j0.020 = j0.045$$

$$y_{12} = \frac{1}{z_{10}} = \frac{1}{0.02 + j0.06} = 5 - j15$$

$$y_{13} = \frac{1}{z_{13}} = \frac{1}{0.08 + j0.24} = 1.25 - j3.75$$

$$y_{23} = \frac{1}{z_{23}} = \frac{1}{0.06 + j0.18} = 1.667 - j5$$

$$Y_{11} = y_{11} + y_{12} + y_{13} = j0.055 + 5 - j15 + 1.25 - j3.75 = 6.25 - j18.695$$

$$Y_{12} = Y_{21} = -y_{12} = -5 + j15$$

$$Y_{13} = Y_{31} = -y_{13} = -1.25 + j3.75$$

$$Y_{22} = y_{22} + y_{21} + y_{23} = j0.050 + 5 - j15 + 1.667 - j5 = 6.667 - j19.95$$

$$Y_{23} = Y_{32} = -y_{23} = -1.667 + j5$$

$$Y_{33} = y_{33} + y_{31} + y_{32} = j0.045 + 1.25 - j3.75 + 1.667 - j5$$
$$= 2.917 - j8.705$$

$$\therefore \qquad Y_{bus} = \begin{bmatrix} Y_{11} & Y_{12} & Y_{13} \\ Y_{21} & Y_{22} & Y_{23} \\ Y_{31} & Y_{32} & Y_{33} \end{bmatrix}$$

$$= \begin{bmatrix} 6.25 - j18.695 & -5 + j15 & -1.25 + j3.75 \\ -5 + j15 & 6.667 - j19.95 & -1.667 + j5 \\ -1.25 + j3.75 & -1.667 + j5 & 2.917 - j8.705 \end{bmatrix}$$

EXAMPLE 1.8 Determine the [Y_{bus}] matrix of the representative power system network diagram shown in Figure 1.23.

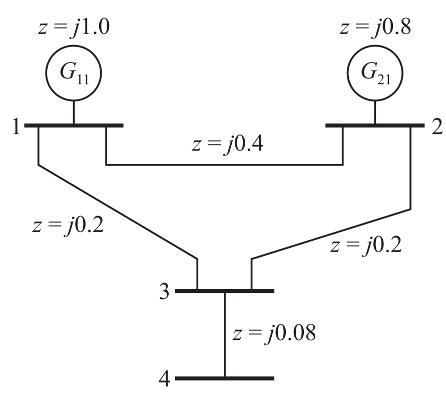


Figure 1.23 Representative power system: Example 1.8.

Solution:

$$y_{11} = \frac{1}{z_{11}} = \frac{1}{j1.0} = -j1.0$$

$$y_{22} = \frac{1}{z_{22}} = \frac{1}{j0.8} = -j1.25$$

$$y_{12} = \frac{1}{z_{12}} = \frac{1}{j0.4} = -j2.5$$

$$y_{13} = \frac{1}{z_{13}} = \frac{1}{j0.2} = -j5$$

$$y_{23} = \frac{1}{z_{23}} = \frac{1}{j0.2} = -j5$$

$$y_{34} = \frac{1}{z_{34}} = \frac{1}{j0.08} = -j12.5$$

$$Y_{11} = y_{10} + y_{12} + y_{13} = -j1.0 - j2.5 - j5 = -j8.5$$

$$Y_{12} = Y_{21} = -y_{12} = j2.5$$

$$Y_{13} = Y_{31} = -y_{13} = j5$$

$$Y_{22} = y_{20} + y_{21} + y_{23} = -j1.25 - j2.5 - j5 = -j8.75$$

$$Y_{23} = Y_{32} = -y_{23} = j5$$

$$Y_{33} = y_{31} + y_{32} + y_{34} = -j5 - j5 - j12.5 = -j22.5$$

$$Y_{44} = y_{43} = -j12.5$$

$$\therefore \qquad Y_{\text{bus}} = \begin{bmatrix} -j8.5 & j2.5 & j5 & 0 \\ j2.5 & -j8.75 & j5 & 0 \\ j5 & j5 & -j22.5 & -j12.5 \\ 0 & 0 & -j12.5 & -j12.5 \end{bmatrix}$$

EXAMPLE 1.9 Determine the $[Y_{\text{bus}}]$ matrix of the representative power system network diagram shown in Figure 1.24.

Verify the result using MATLAB program.

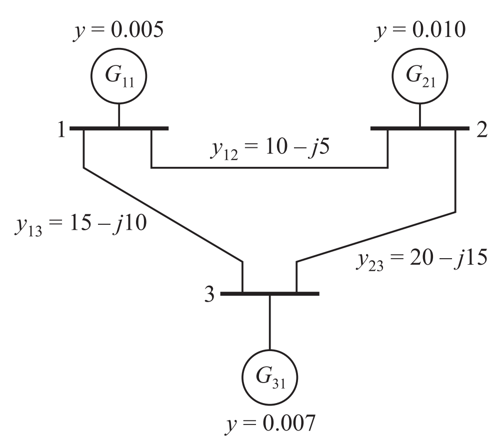


Figure 1.24 Representative power system: Example 1.9.

Solution:

$$y_{11} = 0.005$$
$$y_{22} = 0.010$$
$$y_{33} = 0.007$$
$$y_{12} = 10 - j5$$
$$y_{13} = 15 - j10$$
$$y_{23} = 20 - j15$$
$$Y_{11} = y_{11} + y_{12} + y_{13} = 0.005 + 10 - j5 + 15 - j10 = 25.0050 - j15$$
$$Y_{22} = Y_{21} = -y_{12} = -10 + j5$$
$$Y_{13} = Y_{31} = -y_{13} = -15 + j10$$
$$Y_{22} = y_{22} + y_{21} + y_{23} = 0.010 + 10 - j5 + 20 - j15 = 30.010 - j20$$
$$Y_{23} = Y_{32} = -y_{23} = -20 + j15$$
$$Y_{33} = y_{33} + y_{31} + y_{32} = 0.007 + 15 - j10 + 20 - j15 = 35.0070 - j25$$

$$\therefore \qquad Y_{bus} = \begin{bmatrix} 25.0050 - j15 & -10 + j5 & -15 + j10 \\ -10 + j5 & 30.010 - j20 & -20 + j15 \\ -15 + j10 & -20 + j15 & 35.0070 - j25 \end{bmatrix}$$

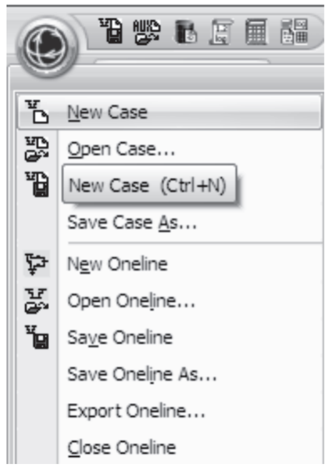
1.7.2 MATLAB Program for [Y_{bus}] Formation Using the Direct Inspection Method

```
clear all;
clc;
close all;
n=input('\n\n Enter the number of buses n=');
b=input('\n\n press your:\n>>1.for impedance  2.for admittance
\n');
if (b==2)
    fprintf('\n\n Enter the admittance value:');
end
if(b==1)
    fprintf('\n\n Enter the impedance value:');
end
for i=1:n
    for j=i+1:n
    a(i,j)=0;
    fprintf('\n Enter the bus %d to bus %d values:',i,j);
    a(i,j)=input('');
    if(b~=2&a(i,j)~=0)
      a(i,j)=1/a(i,j);
```

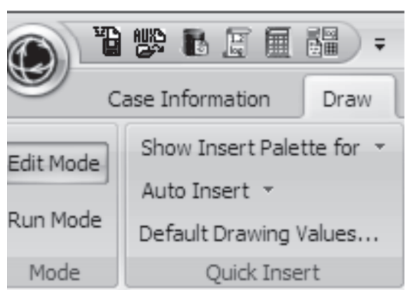
```
     end
     a(j,i)=a(i,j);
  end
end
d=input('\n\n   Is there any admittance value in source?\n
Press "1" for yes and "2" for no:');
if (d==1)
     for i=1:n
       fprintf('\n Enter for bus %d:',i);
       a(i,i)=input('');
     end
end
for i=1:n
       for j=1:n
         if(i~=j)
           a(i,i)=a(i,i)+a(i,j);
           y(i,j)=-a(i,j);
         end
       end
       y(i,i)=a(i,i);
end
fprintf('\n\n Y bus=:>>\n\n');
display(y);

output:
——--

Enter the number of buses n=3
     press your:
>>1.for impedance   2.for admittance
     2
     Enter the admittance value:
     Enter the bus 1 to bus 2 values:10-5j
     Enter the bus 1 to bus 3 values:15-10j
     Enter the bus 2 to bus 3 values:20-15j
       Is there any admittance value in source?
     Press "1" for yes and "2" for no:1
     Enter for bus 1:.005
     Enter for bus 2:.010
     Enter for bus 3:.007
     Y bus=:>>
y =
     25.0050 - 15.0000i - 10.0000 + 5.0000i - 15.0000 + 10.0000i
     -10.0000 + 5.0000i  30.0100 - 20.0000i - 20.0000 + 15.0000i
     -15.0000 + 10.0000i - 20.0000 + 15.0000i  35.0070 - 25.0000i
```

1.7.3 Formation of [Y_{bus}] Using Power World Simulator

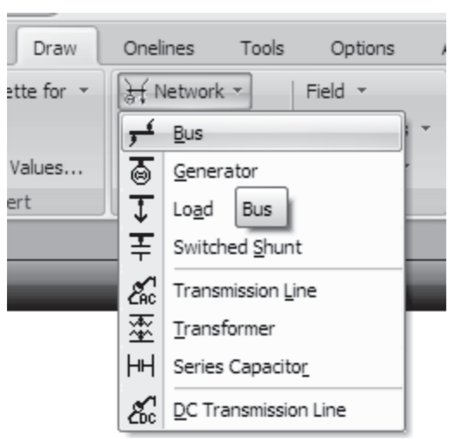
1. Start the Power World Simulator by clicking the icon
2. Go to file and click New Case and new window will open



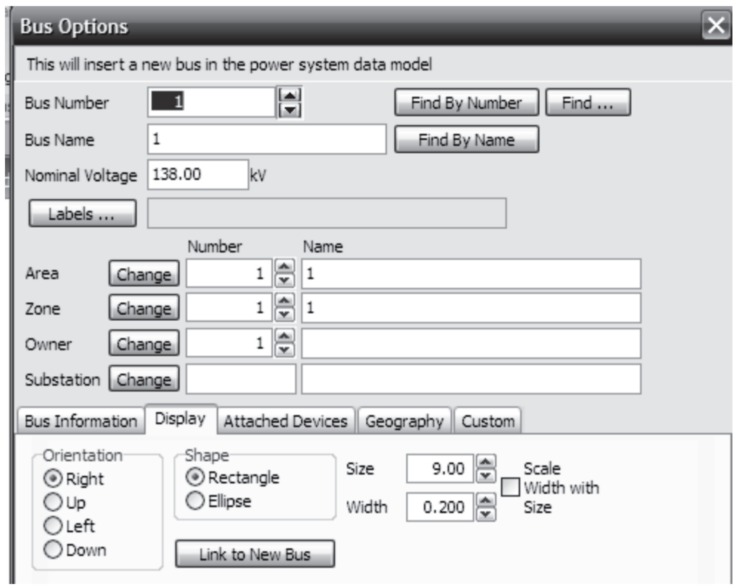
3. Select the draw tab which is on the top.



4. Using the option Network in the menu draw the one line diagram.

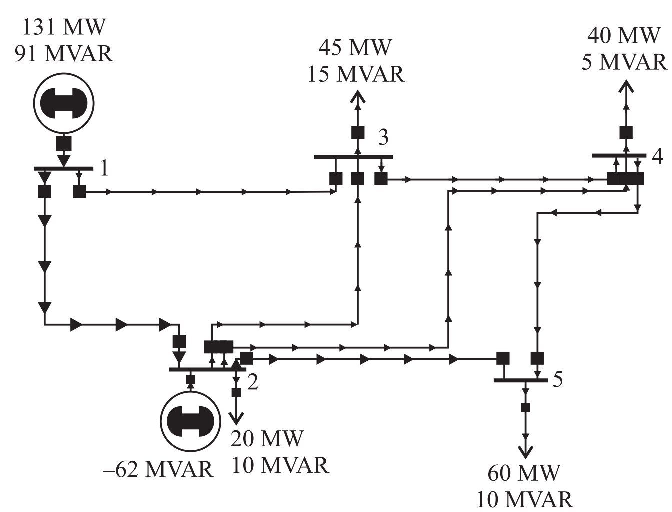


5. The input data for Buses, Generators, Loads, Transmission lines can be given by selecting the bus in the Network menu and left clicking the mouse in the new window opened.

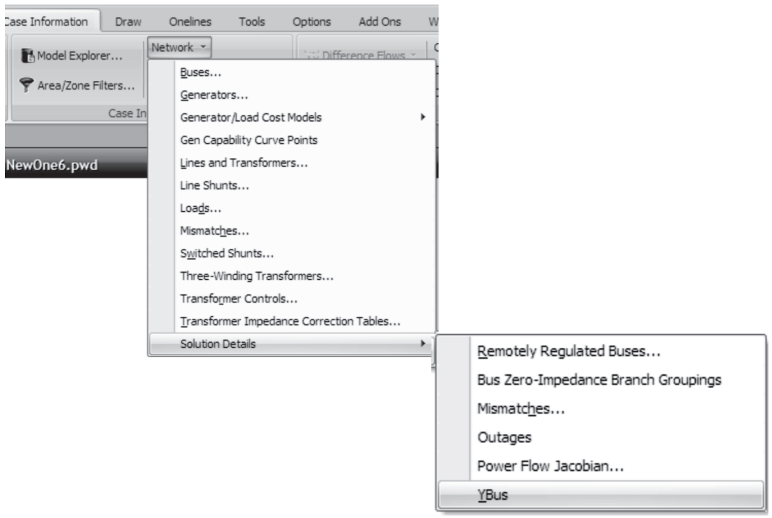


6. The bus can be shown as

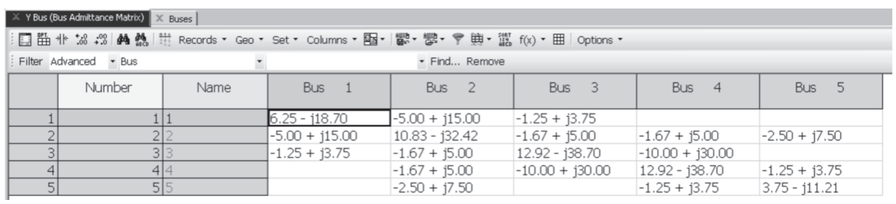
7. The same procedure is followed to draw generators, loads, etc.
8. Then the one line diagram for the 5 bus system can be obtained as



9. Go to Case information tab in the menu and select Solution details in the network tab, and select Y_{Bus}.



10. Then the matrix $[Y_{bus}]$ will be displayed as

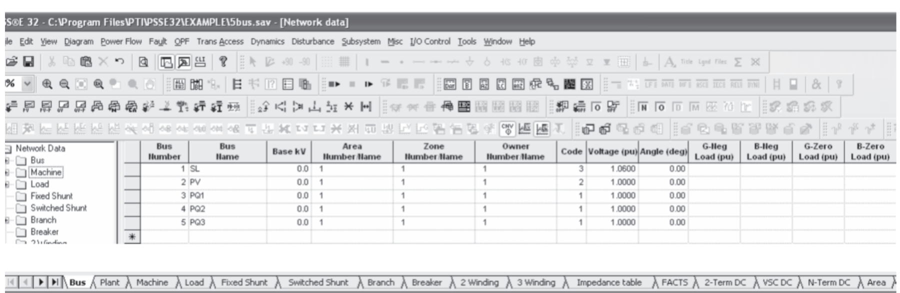


	Number	Name	Bus 1	Bus 2	Bus 3	Bus 4	Bus 5
1	1	1	6.25 - j18.70	-5.00 + j15.00	-1.25 + j3.75		
2	2	2	-5.00 + j15.00	10.83 - j32.42	-1.67 + j5.00	-1.67 + j5.00	-2.50 + j7.50
3	3	3	-1.25 + j3.75	-1.67 + j5.00	12.92 - j38.70	-10.00 + j30.00	
4	4	4		-1.67 + j5.00	-10.00 + j30.00	12.92 - j38.70	-1.25 + j3.75
5	5	5		-2.50 + j7.50		-1.25 + j3.75	3.75 - j11.21

1.7.4 Formation of $[Y_{bus}]$ Using PSS/E

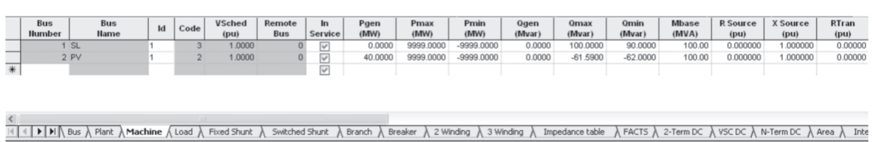
Let us consider a sample of 5 a bus system.

Bus data:



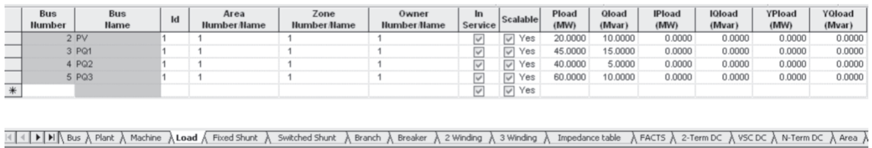
Bus Number	Bus Name	Base kV	Area Number/Name	Zone Number/Name	Owner Number/Name	Code	Voltage (pu)	Angle (deg)	G-Neg Load (pu)	B-Neg Load (pu)	G-Zero Load (pu)	B-Zero Load (pu)
1	SL	0.0	1	1	1	3	1.0600	0.00				
2	PV	0.0	1	1	1	2	1.0000	0.00				
3	PQ1	0.0	1	1	1	1	1.0000	0.00				
4	PQ2	0.0	1	1	1	1	1.0000	0.00				
5	PQ3	0.0	1	1	1	1	1.0000	0.00				

Machine data:



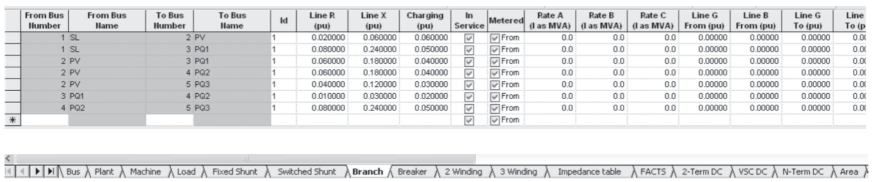
Bus Number	Bus Name	Id	Code	VSched (pu)	Remote Bus	In Service	Pgen (MW)	Pmax (MW)	Pmin (MW)	Qgen (Mvar)	Qmax (Mvar)	Qmin (Mvar)	Mbase (MVA)	R Source (pu)	X Source (pu)	RTran (pu)
1	SL	1	3	1.0000	0	☑	0.0000	9999.0000	-9999.0000	0.0000	100.0000	90.0000	100.00	0.000000	1.000000	0.00000
2	PV	1	2	1.0000	0	☑	40.0000	9999.0000	-9999.0000	0.0000	-61.5900	-62.0000	100.00	0.000000	1.000000	0.00000

Load data:



Bus Number	Bus Name	Id	Area Number/Name	Zone Number/Name	Owner Number/Name	In Service	Scalable	Pload (MW)	Qload (Mvar)	IPload (MW)	IQload (Mvar)	YPload (MW)	YQload (Mvar)
2	PV	1	1	1	1	✓	✓ Yes	20.0000	10.0000	0.0000	0.0000	0.0000	0.0000
3	PQ1	1	1	1	1	✓	✓ Yes	45.0000	15.0000	0.0000	0.0000	0.0000	0.0000
4	PQ2	1	1	1	1	✓	✓ Yes	40.0000	5.0000	0.0000	0.0000	0.0000	0.0000
5	PQ3	1	1	1	1	✓	✓ Yes	60.0000	10.0000	0.0000	0.0000	0.0000	0.0000
*						✓	✓ Yes						

Branch data:

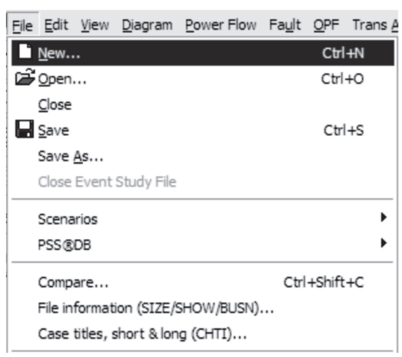


From Bus Number	From Bus Name	To Bus Number	To Bus Name	Id	Line R (pu)	Line X (pu)	Charging (pu)	In Service	Metered	Rate A (I as MVA)	Rate B (I as MVA)	Rate C (I as MVA)	Line G From (pu)	Line B From (pu)	Line G To (pu)	Line To (p
1	SL	2	PV	1	0.020000	0.060000	0.060000	✓	✓ From	0.0	0.0	0.0	0.00000	0.00000	0.00000	0.0(
1	SL	3	PQ1	1	0.080000	0.240000	0.050000	✓	✓ From	0.0	0.0	0.0	0.00000	0.00000	0.00000	0.0(
2	PV	3	PQ1	1	0.060000	0.180000	0.040000	✓	✓ From	0.0	0.0	0.0	0.00000	0.00000	0.00000	0.0(
2	PV	4	PQ2	1	0.060000	0.180000	0.040000	✓	✓ From	0.0	0.0	0.0	0.00000	0.00000	0.00000	0.0(
2	PV	5	PQ3	1	0.040000	0.120000	0.030000	✓	✓ From	0.0	0.0	0.0	0.00000	0.00000	0.00000	0.0(
3	PQ1	4	PQ2	1	0.010000	0.030000	0.020000	✓	✓ From	0.0	0.0	0.0	0.00000	0.00000	0.00000	0.0(
4	PQ2	5	PQ3	1	0.080000	0.240000	0.050000	✓	✓ From	0.0	0.0	0.0	0.00000	0.00000	0.00000	0.0(
*								✓	✓ From							

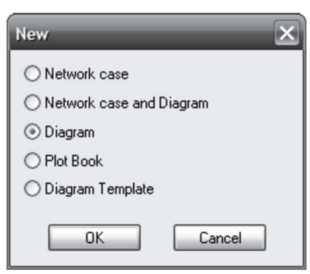
In the network data we have to specify bus data, machine data, load data, fixed shunt data, etc., and save as six.sav file

Using slider file to create one line diagram

1. First we need to load a file as shown above.
2. Open **PSS/E**.
3. Go to file, and click new as shown below.

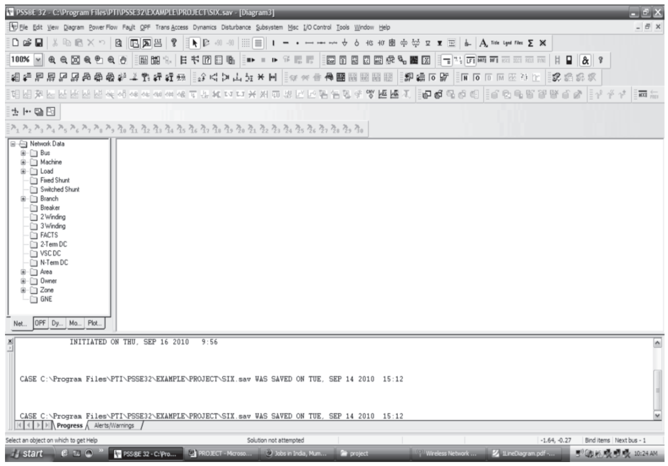


4. After clicking New the following will be displayed:

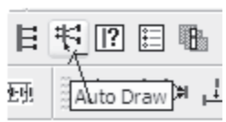


select **Diagram** and click **OK**.

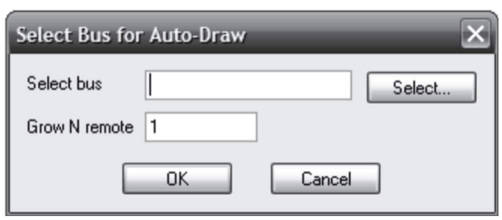
5. A new diagram will be displayed.



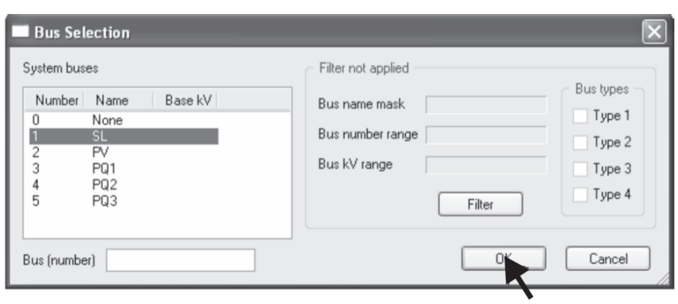
6. A bus can be imported from the data (*sav) file into the diagram. For this select **Auto Draw** function on the toolbar.



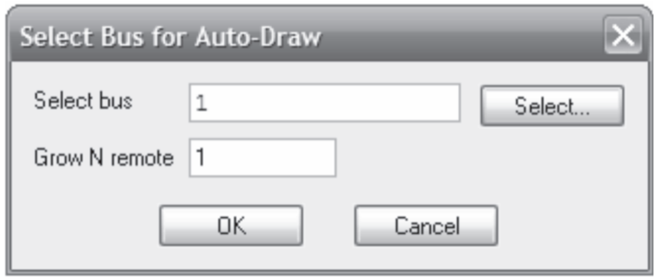
7. Click on an open place in a blank diagram, the following will be displayed:



8. Next click **Select** in the select bus window or type the desired bus number from the *sav case file (In this case bus number 1 from six.sav file)

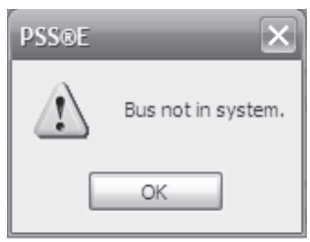


Click **OK** to return to the select bus window, which will show the bus selected.



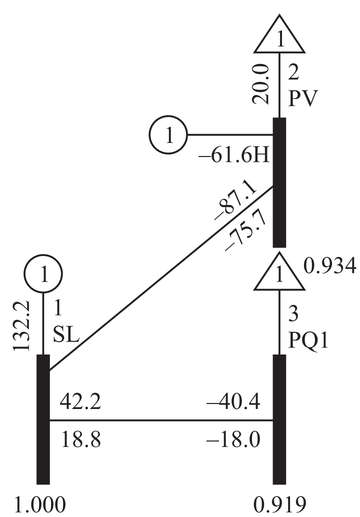
Click **OK.**

Note: As previously mentioned the bus can also be typed manually **to** skip the select bus window. If a bus number is not in a *.sav file, the following error message will be displayed.



click **OK** and enter any one of the bus numbers from *.sav file.

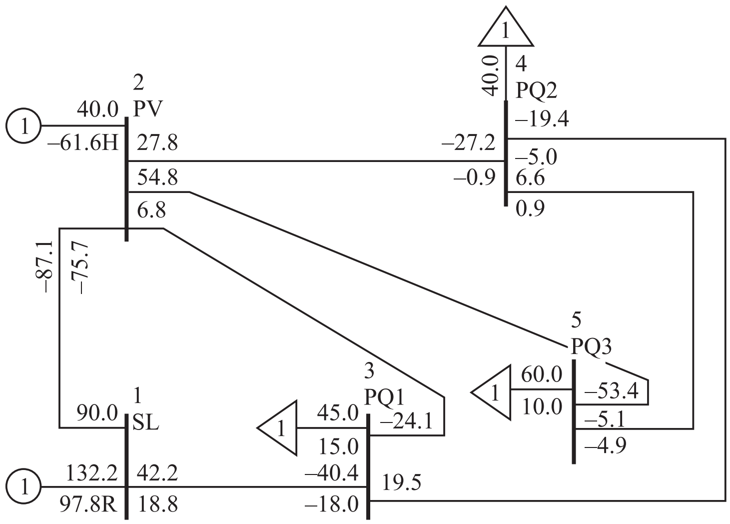
9. Select **bus number 1** and all devices connected to it are displayed as shown below:



Select the **pointer** from the toolbar to exit out of the **AutoDraw** function.

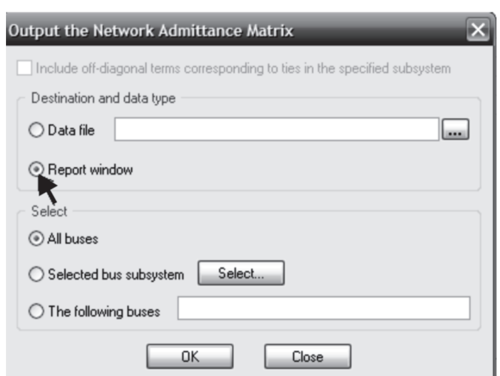
This will help to rearrange the one-line diagram so that no wires or ratings are overlapping.

10. Select all the buses one by one and try to rearrange them, the final diagram will be shown as:



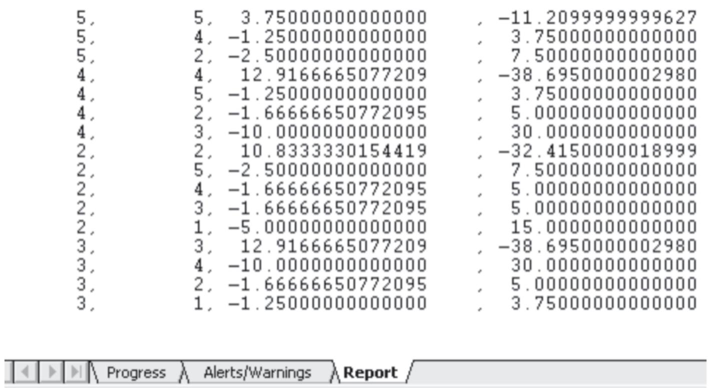
Formation of [Y_{bus}] using PSS/E

1. Load a file from a *.sav file.
2. Go to file menu and select **EXPORT** and then select **NETWORK ADMITTANCE MATRIX.**
3. Admittance matrix window will be displayed as



Select **REPORT WINDOW** and click **OK**.

4. [Y_{bus}] matrix can be obtained in the report window as:



```
5,        5,  3.75000000000000    , -11.2099999999627
5,        4, -1.25000000000000    ,  3.75000000000000
5,        2, -2.50000000000000    ,  7.50000000000000
4,        4,  12.9166665077209    , -38.6950000002980
4,        5, -1.25000000000000    ,  3.75000000000000
4,        2, -1.66666650772095    ,  5.00000000000000
4,        3, -10.0000000000000    ,  30.0000000000000
2,        2,  10.8333330154419    , -32.4150000018999
2,        5, -2.50000000000000    ,  7.50000000000000
2,        4, -1.66666650772095    ,  5.00000000000000
2,        3, -1.66666650772095    ,  5.00000000000000
2,        1, -5.00000000000000    ,  15.0000000000000
3,        3,  12.9166665077209    , -38.6950000002980
3,        4, -10.0000000000000    ,  30.0000000000000
3,        2, -1.66666650772095    ,  5.00000000000000
3,        1, -1.25000000000000    ,  3.75000000000000
```


1.7.5 Singular Transformation Method

The matrix [Y_{bus}] can be determined by using the bus incidence matrix A and the related variable parameters of the primitive network quantities of the interconnected network.

From the primitive network equation,

$$i + j = [y]\overline{V} \tag{1.46}$$

Multiplying both sides of Eq. (1.46) by A^T, we get

$$A^T i + A^T j = A^T [y]\overline{V} \tag{1.47}$$

According to KCL, the algebraic sum of the currents meeting at any node is equal to zero. Thus,

$$A^T i = 0 \tag{1.48}$$

Similarly, $A^T j$ is equal to the sum of the current sources of an element incident at a node. It is a column vector. Thus,

$$A^T j = I_{bus} \tag{1.49}$$

Substituting Eqs. (1.48) and (1.49) in Eq. (1.47), we have

$$I_{bus} = A^T [y]\overline{V} \tag{1.50}$$

Power into the network is $(\overline{I}^*_{bus})^T \overline{E}_{bus}$ and equal to the sum of powers in the primitive network, i.e. $(j^*)^T \overline{V}$

The power in the primitive network

$$(\overline{I}^*_{bus})^T \overline{E}_{bus} = (j^*)^T \overline{V} \tag{1.51}$$

Taking conjugate transpose of Eq. (1.49),

$$(\overline{I}^*_{bus})^T = (A^T)^{*T} (j^*)^T$$

But A is a real matrix, so $A^* = A$

From the matrix property, $(A^T)^T = A$

Applying these two conditions in Eq. (1.51),

$$(\bar{I}^*_{\text{bus}})^T = A(j^*)^T \tag{1.52}$$

Substituting Eq. (1.52) into Eq. (1.51),

$$A(j^*)^T \bar{E}_{\text{bus}} = (j^*)^T \bar{V} \tag{1.53}$$

$$A\bar{E}_{\text{bus}} = \bar{V} \tag{1.54}$$

Substituting $\bar{V}$ from Eq. (1.54) into Eq. (1.50),

$$\bar{I}_{\text{bus}} = A^T[y]A\bar{E}_{\text{bus}} \tag{1.55}$$

$$\therefore \qquad \frac{\bar{I}_{\text{bus}}}{\bar{E}_{\text{bus}}} = Y_{\text{bus}} = A^T[y]A \tag{1.56}$$

EXAMPLE 1.10 Form the matrix $[Y_{\text{bus}}]$ using the singular transformation method for the system shown in Figure 1.24. The impedance data is given in the Table 1.2. Take (1) as the reference node.

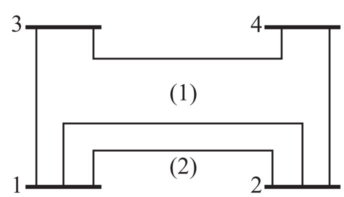


Figure 1.24 Sample power system of Example 1.10.

Table 1.2 Impedance data of Example 1.10

Element No.	***Self***	
	Bus code	***Impedance***
1	1–2 (1)	0.6
2	1–3	0.5
3	3–4	0.5
4	1–2 (2)	0.4
5	2–4	0.2

Solution: The oriented graph of the system is shown in Figure 1.25.

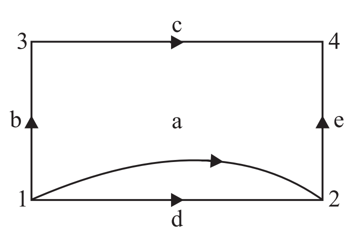


Figure 1.25 Oriented graph of the system shown in Figure 1.24.

Take (1) as the reference node,

$$\overline{A} = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix}$$

Hence,

$$A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 0 \\ 1 & 0 & -1 \end{bmatrix}$$

and

$$A^T = \begin{bmatrix} -1 & 0 & 0 & -1 & 1 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & -1 \end{bmatrix}$$

From the given impedance data,

$$Z_{\text{primitive}} = \begin{bmatrix} j0.6 & 0 & 0 & 0 & 0 \\ 0 & j0.5 & 0 & 0 & 0 \\ 0 & 0 & j0.5 & 0 & 0 \\ 0 & 0 & 0 & j0.4 & 0 \\ 0 & 0 & 0 & 0 & j0.2 \end{bmatrix}$$

Therefore,

$$Y_{\text{primitive}} = [Z_{\text{primitive}}]^{-1} = \begin{bmatrix} -j1.667 & 0 & 0 & 0 & 0 \\ 0 & -j2 & 0 & 0 & 0 \\ 0 & 0 & -j2 & 0 & 0 \\ 0 & 0 & 0 & -j2.5 & 0 \\ 0 & 0 & 0 & 0 & -j5 \end{bmatrix}$$

Hence

$$Y_{\text{bus}} = [A^T]\ [Y_{\text{primitive}}]\ [A]$$

Therefore,

$$Y_{\text{bus}} = \begin{bmatrix} -j9.167 & 0 & j5 \\ 0 & -j4 & j2 \\ j5 & j2 & -j7 \end{bmatrix}$$

EXAMPLE 1.11 Form the matrix [Y_{bus}] using the singular transformation method for the system shown in Figure 1.26. The impedance data is given in Table 1.3. Take (1) as the reference node.

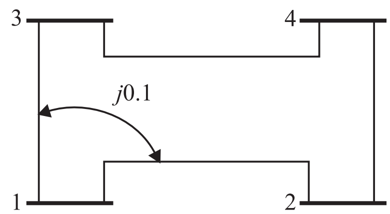


Figure 1.26 Sample power system of Example 1.11.

Table 1.3 Impedance data of Example 1.11

Element no.	*Self*		*Mutual*	
	Bus code	***Impedance***	***Bus code***	***Impedance***
1	1–2	0.5		
2	1–3	0.6	1–2	0.1
3	3–4	0.4		
4	2–4	0.3		

Solution: The oriented graph to the system is shown in Figure 1.27.

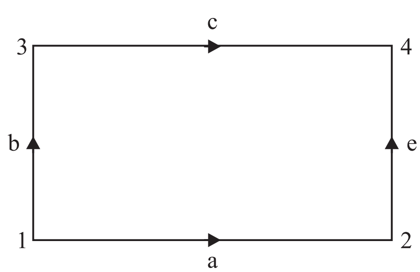


Figure 1.27 Oriented graph of the system shown in Figure 1.26.

Take (1) as the reference node,

$$\overline{A} = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & -1 \end{bmatrix}$$

Hence,

$$A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 1 & -1 \\ 1 & 0 & -1 \end{bmatrix}$$

and

$$A^T = \begin{bmatrix} -1 & 0 & 0 & 1 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & -1 \end{bmatrix}$$

From the given impedance data,

$$Z_{\text{primitive}} = \begin{bmatrix} j0.5 & j0.1 & 0 & 0 \\ j0.1 & j0.6 & 0 & 0 \\ 0 & 0 & j0.4 & 0 \\ 0 & 0 & 0 & j0.3 \end{bmatrix}$$

Therefore,

$$Y_{\text{primitive}} = [Z_{\text{primitive}}]^{-1}$$

Consider the 2 × 2 matrix

$$\begin{bmatrix} j0.5 & j0.1 \\ j0.1 & j0.6 \end{bmatrix}^{-1} = \frac{1}{-0.29}\begin{bmatrix} j0.6 & -j0.1 \\ -j0.1 & j0.5 \end{bmatrix} = \begin{bmatrix} -j2.0689 & j0.3448 \\ j0.3448 & -j1.724 \end{bmatrix}$$

Hence,

$$Y_{\text{primitive}} = \begin{bmatrix} -j2.0689 & j0.3448 & 0 & 0 \\ j0.3448 & -j1.724 & 0 & 0 \\ 0 & 0 & -j2.5 & 0 \\ 0 & 0 & 0 & -j3.333 \end{bmatrix}$$

Now,

$$Y_{\text{bus}} = [A^T]\,[Y_{\text{primitive}}]\,[A]$$

Therefore,

$$Y_{\text{bus}} = \begin{bmatrix} -j5.4019 & j0.3448 & j3.333 \\ j0.3448 & -j4.224 & j2.5 \\ j3.333 & j2.5 & -j5.833 \end{bmatrix}$$

1.8 Solution Technique

The mathematical model of power system networks used for the purposes of load flow studies, short circuit studies and stabilities studies, is a set of linear or nonlinear algebraic equations or differential equations or both. But since the digital computer performs only four basic operations of addition, subtraction, multiplication and division, hence in order to solve these mathematical equations on the digital computer, it is necessary to transform these linear or nonlinear algebraic and differential equations to a set of four operations of addition, subtraction, multiplication and division with the help of numerical methods.

Solution of algebraic equations

There is a different numerical technique for the solution of algebraic equations. These algebraic equations can be expressed in the following form.

$$\left.\begin{aligned} f_1(x_1, x_2, \ldots, x_n) &= y_1 \\ f_2(x_1, x_2, \ldots, x_n) &= y_2 \\ \vdots \qquad\qquad & \quad \vdots \\ f_n(x_1, x_2, \ldots, x_n) &= y_n \end{aligned}\right\} \tag{1.57}$$

Here f_i are the functions relating the unknown variables x_i, with the known constants y_i. If any one of the f_i is nonlinear, the above algebraic equations form a set of nonlinear algebraic equations (equations involving power terms or the product of variables x_i). However, if all of the f_is are linear, then the above algebraic equations form a set of linear algebraic equations.

The linear algebraic equations can be expressed as follows:

$$[A]X = Y \tag{1.58}$$

where,

$[A]$ = the coefficient matrix of the physical system.
X = a column vector of unknowns
Y = a column vector of known constants.

Equation (1.58) can then be expressed as

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} \tag{1.59}$$

In order to choose a numerical method to solve a set of algebraic equations on the digital computer, the following points must be considered:

(i) Number of steps needed to obtain the solution, i.e. speed at which the solution is obtained.
(ii) Resultant accuracy
(iii) Computer memory limitations.

The numerical techniques to solve a set of linear algebraic equations can be broadly classified into two main headings, namely.

(i) Direct method or exact method
(ii) Iterative technique

In the case of *direct method*, the solution can be obtained in a distinct number of steps. However, the number of steps required to obtain the solution depends upon the problem size, i.e. the order of the coefficient matrix [A] and the numerical method used. Hence before use, it is possible to compare different direct methods since the number of steps required to obtain the solution is known in advance. The only error is the loss of significant digits which is not bounded. The solution will be far from nominal solution because the round-off error goes on getting accumulated after each step. The round-off error results due to the subtraction or division by two numbers which are nearly equal. If the round-off error is bounded, the solution obtained will be nearly exact. This is why this method is also known as *exact method*.

Direct methods are:

(i) Cramer's rule
(ii) Gauss elimination method and
(iii) Gauss–Jordan elimination method

In the case of *iterative techniques*, the solution is obtained in an orderly fashion starting from its initial approximate solution, i.e. initial guess. Here we start with the initial approximate solution. Thus the rate of convergence, i.e. the number of steps needed to obtain solution depends upon the initial guess, problem size (number of equations) and the iterative the techniques used. Thus depending upon these factors, the approximate solution may converge to the nominal solution, diverge or oscillate about the nominal solution. The round-off error in this case goes on getting corrected in each step, .i.e. in each iteration.

The iterative techniques are:

(i) Gauss iterative technique
(ii) Gauss–Seidel iterative technique
(iii) Newton–Raphson method.

1.8.1 Sparse Matrix Techniques for Large-Scale Power Systems

The term "sparsity" is used to indicate the relative absence of certain problem interconnections. Mathematically, we can define:

Given a finite discrete sample space Ω *and a nonempty set of sample S such that the cardinality* $|S|$ *of S is small compared to cardinality* $|\Omega|$ *of* Ω, i.e. $|S| \ll |\Omega|$ *is then said to be sparse with respect to S.*

Let

$$n = |S|$$

$$N = |\Omega|$$

The efficient handling of sparse matrices is at the heart of almost every non-trivial power systems computational problem. Engineering problems have two stages. The first stage is an understanding and formulation of the problem in precise terms. The second stage is the solution of the problem. Many problems in power systems result in formulations that require the use of large sparse matrices. The well-known problems that fit into this category include three classic problems: power flow, short circuit, and transient stability. To this list we can also add numerous other important system problems: electromagnetic transients, economic dispatch, optimal power flows, state estimation, and contingency studies, just to name a few. In addition, the problems that require finite element or finite-difference methods for their solution invariably end up in mathematical formulations where sparse matrices are involved. Sparse matrices are also important for electronic circuits and numerous other engineering problems. We describe sparse matrices primarily from the perspective of power system network, though most of the ideas and results are readily applicable to more general sparse matrix problems.

The key idea behind sparse matrices is computational complexity. Storage requirements for a full matrix increase as order n^2. Computational requirements for many full matrix operations increase as order n^3. By the very definition of a sparse matrix, the storage and computational requirements for most sparse matrix operations increase only linearly or close to linearly. Faster computers will help solve larger problems, but unless the speedups are of order n^3 they will not keep pace with the advantages attainable from sparsity in the larger problems of the future. Thus, sparse matrices have become and will continue to be important. This introduction retraces some of the principal steps in the progress on sparse matrix theory.

Sparse system

Most of the discrete sample spaces are sparsely populated. Natural occurrences of sparsity are wide ranging. The following is a partial list of the sparse systems:

(i) Networks of all kinds such as electric power, electronics and communications, hydraulics, etc.
(ii) Space trusses and frames of structures.
(iii) Roads, highways and airways connecting all the important cities of the world.
(iv) Street connections among intersections within a city.
(v) Matrices associated with algebraic equations resulting from different methods in the solution of differential equations.
(vi) Matrices arising in the discrete analysis of continuous functions.

1.8.2 Optimally Ordered Triangular Factorization

Usually, the objective in the matrix analysis of networks is to obtain the inverse of the matrix of coefficients of a system of simultaneous linear network equations. However, for large sparse systems such as those which occur in many network problems, the use of the inverse is very inefficient. In general, the matrix of the equations formed from the given conditions of a network problem is sparse, whereas its inverse is full. By means of an appropriately ordered triangular decomposition, the inverse of a sparse matrix can be expressed as a product of sparse matrix factors, thereby gaining an advantage in computational speed, storage, and reduction of round-off error.

The method consists of two parts:

1. Recording the operations of triangular decomposition of a matrix such that repeated direct solutions based on the matrix can be obtained without repeating the triangularization
2. Ordering the operations that tends to conserve the sparsity of the original system.

Either part can be applied independently, but the greatest benefit is obtained from the combined application of both parts. The first part can be applied to any matrix. The application of the second part, i.e. ordering to conserve sparsity, is limited to sparse matrices in which the pattern of nonzero elements is symmetric and for which an arbitrary order of decomposition does not adversely affect numerical accuracy. Such matrices are usually characterized by a strong diagonal, and ordering to conserve sparsity increases the accuracy of the decomposition. A large class of network problems fulfils this condition. Generally it is not worth considering optimal ordering unless at least 80 per cent of the matrix elements are zero.

Factored direct solutions

The first part of the above method shows how to derive an array of numbers from a nonsingular matrix A that can be used to obtain the effects of any or all of the following: A, A^{-1}, A^{T}, (A^{T-1}), and certain two-way hybrid combinations of these matrices. The method is applicable to any nonsingular matrix, real or complex, sparse or full, symmetric or non-symmetric. This method is also applicable to mesh equations. Its greatest advantage is realized in problems involving large sparse matrices. The basic scheme is first presented for the most general case, a full non-symmetric matrix. Symmetry is then treated as a special case.

1.8.3 Triangular Decomposition—Gaussian Elimination

Triangular decomposition of a matrix by Gaussian elimination is described in many books on matrix analysis. Ordinarily, the decomposition is accomplished by the elimination of elements below the main diagonal in successive columns. For the purpose of computer programming for a sparse matrix, it is usually much more efficient to eliminate elements by successive rows. The development is based on the equation

$$AX = b \tag{1.60}$$

where A is a nonsingular matrix, X is a column vector of unknowns, and b is a known vector with at least one nonzero element, i.e.

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

In the computer algorithm, A is augmented by b as shown in Eq. (1.61) for an nth-order system.

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & & & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} & b_n \end{bmatrix} \tag{1.61}$$

The first step involves the division of the elements of the first row by a_{11}, as shown in Eq. (1.62).

$$\begin{aligned} a_{1j}^{(1)} &= \left(\frac{1}{a_{11}}\right) a_{1j} \qquad j = 2,\ n \\ b_1^{(1)} &= \left(\frac{1}{a_{11}}\right) b_1 \end{aligned} \tag{1.62}$$

The superscripts indicate the order of the derived system. The second step, as shown in Eqs. (1.63a) and (1.63b), involves elimination of a_{21}, from the second row by a linear combination with the derived first row, and then dividing the remaining derived elements of the second row by its derived diagonal element.

$$\begin{bmatrix} 1 & a_{12}^{(1)} & a_{13}^{(1)} & \cdots & a_{1n}^{(1)} & b_1^{(1)} \\ 0 & 1 & a_{23}^{(2)} & \cdots & a_{2n}^{(2)} & b_2^{(2)} \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} & b_n \end{bmatrix} \tag{1.63a}$$

$$\begin{aligned} a_{2j}^{(1)} &= a_{2j} - a_{21} a_{1j}^{(1)} \qquad j = 2,\ n \\ b_2^{(1)} &= b_2 - a_{21} b_1^{(1)} \\ a_{2j}^{(2)} &= \left(\frac{1}{a_{22}^{(1)}}\right) a_{2j}^{(1)} \qquad j = 3,\ n \\ b_2^{(2)} &= \left(\frac{1}{a_{22}^{(1)}}\right) b_2^{(1)} \end{aligned} \tag{1.63b}$$

The third step, as shown in Eqs. (1.64a) and (1.64b), involves elimination of elements to the left of the diagonal of the third row and dividing the remaining derived elements of the row by the derived diagonal element.

$$\begin{bmatrix} 1 & a_{12}^{(1)} & a_{13}^{(1)} & a_{14}^{(1)} & \cdots & a_{1n}^{(1)} & b_1^{(1)} \\ 0 & 1 & a_{23}^{(2)} & a_{24}^{(2)} & \cdots & a_{2n}^{(2)} & b_2^{(1)} \\ 0 & 0 & 1 & a_{34}^{(3)} & \cdots & a_{3n}^{(3)} & b_3^{(1)} \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & a_{n4} & \cdots & a_{nn} & b_n \end{bmatrix} \tag{1.64a}$$

$$
\begin{aligned}
a_{3j}^{(1)} &= a_{3j} - a_{31}a_{1j}^{(1)} \qquad j = 2,\ n \\
b_3^{(1)} &= b_3 - a_{31}b_1^{(1)} \\
a_{3j}^{(2)} &= a_{3j}^{(1)} - a_{32}^{(1)}\, a_{2j}^{(2)} \qquad j = 3,\ n \\
b_3^{(2)} &= b_3^{(1)} - a_{32}^{(1)}\, b_2^{(2)} \\
a_{3j}^{(3)} &= \left(\frac{1}{a_{33}^{(2)}}\right) a_{3j}^{(2)} \qquad j = 4,\ n \\
b_3^{(3)} &= \left(\frac{1}{a_{33}^{(2)}}\right) b_3^{(2)}
\end{aligned} \tag{1.64b}
$$

Proceeding in this manner the nth derived system is obtained as shown below:

$$
\begin{bmatrix}
1 & a_{12}^{(1)} & \cdot & \cdot & \cdot & a_{1n}^{(1)} & b_1^{(1)} \\
 & 1 & \cdot & \cdot & \cdot & a_{2n}^{(2)} & b_2^{(1)} \\
 & & \cdot & \cdot & \cdot & \cdot & \cdot \\
 & & & \cdot & \cdot & \cdot & \cdot \\
 & & & & & 1 & b_n^{(n)}
\end{bmatrix} \tag{1.65}
$$

It should be noted that at the end of the kth step, work on rows 1 to k gets completed and rows $k + 1$ to n have not yet entered the process in any way.

The solution can now be obtained by back substitution.

$$
\begin{aligned}
x_n &= b_n^{(n)} \\
x_{n-1} &= b_{n-1}^{(n-1)} - a_{n-1,n}^{(n-1)} \cdot x_n \\
x_i &= b_i^{(i)} - \sum_{j=i+1}^{n} a_{i,j}^{(i)} \cdot x_j
\end{aligned} \tag{1.66}
$$

In programming, the x_i's replace the b_i's one-by-one as they are computed, starting with x_n, and working back to x_1. When A is full and n is large, it can be shown that the number of multiplication–addition operations for triangular decomposition is approximately $1/3n^3$ compared with n^3 for inversion.

It can be easily verified that triangularization in the same order by columns instead of rows would have produced identically the same result. Each eliminated element $a_{ij}^{(j-1)}$, $i > j$, would have been the same and the number of operations would have been the same. The back substitution also could have been accomplished by columns instead of rows in the same number of operations.

EXAMPLE 1.12 Solve the following equations using the Gauss elimination method:

$$
\begin{aligned}
2x_1 + x_2 + 3x_3 &= 6 \\
2x_1 + 3x_2 + 4x_3 &= 9 \\
3x_1 + 4x_2 + 7x_3 &= 14
\end{aligned}
$$

Solution: In the matrix form

$$\begin{bmatrix} 2 & 1 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 7 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 6 \\ 9 \\ 14 \end{bmatrix}$$

$$\therefore \qquad A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} 2 & 1 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 7 \end{bmatrix}$$

$$\text{Augmented matrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & b_1 \\ a_{21} & a_{22} & a_{23} & b_2 \\ a_{31} & a_{32} & a_{33} & b_3 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 3 & 6 \\ 2 & 3 & 4 & 9 \\ 3 & 4 & 7 & 14 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & a_{12}^{(1)} & a_{13}^{(1)} & b_1^{(1)} \\ & 1 & a_{23}^{(2)} & b_2^{(2)} \\ & & 1 & b_3^{(3)} \end{bmatrix}$$

The first step is to divide the elements of the first row by a_{11},

$$a_{1j}^{(1)} = \left(\frac{1}{a_{11}}\right)a_{1j} \qquad j = 2,\ n$$

$$j = 2;\ a_{12}^{(1)} = \left(\frac{1}{a_{11}}\right)a_{12} = \left(\frac{1}{2}\right)1 = 0.5$$

$$j = 3;\ a_{13}^{(1)} = \left(\frac{1}{a_{11}}\right)a_{13} = \left(\frac{1}{2}\right)3 = 1.5$$

$$b_1^{(1)} = \left(\frac{1}{a_{11}}\right)b_1 = \left(\frac{1}{2}\right)\times 6 = 3$$

The second step is to eliminate a_{21}, from the second row by linear combination with the derived first row, and then to divide the remaining derived elements of the second row by its derived diagonal element.

$$a_{2j}^{(1)} = a_{2j} - a_{21}a_{1j}^{(1)} \qquad j = 2,\ n$$

$$j = 2;\ a_{22}^{(1)} = a_{22} - a_{21}a_{12}^{(1)} = 3 - 2\times 0.5 = 2$$

$$j = 3;\ a_{23}^{(1)} = a_{23} - a_{21}a_{13}^{(1)} = 4 - 2\times 1.5 = 1$$

$$b_2^{(1)} = b_2 - a_{21}b_1^{(1)} = 9 - 2 \times 3 = 3$$

$$a_{2j}^{(2)} = \left(\frac{1}{a_{22}^{(1)}}\right)a_{2j}^{(1)} \qquad j = 3,\ n$$

$$j = 3;\ a_{23}^{(2)} = \left(\frac{1}{a_{22}^{(1)}}\right)a_{23}^{(1)} = \left(\frac{1}{2}\right)\times 1 = 0.5$$

$$b_2^{(2)} = \left(\frac{1}{a_{22}^{(1)}}\right)b_2^{(1)} = \left(\frac{1}{2}\right)\times 3 = 1.5$$

The third step is to eliminate the elements to the left of the diagonal of the third row and to divide the remaining derived elements of the row by the derived diagonal element.

$$a_{3j}^{(1)} = a_{3j} - a_{31}a_{1j}^{(1)} \qquad j = 2,\ n$$

$$j = 2;\ a_{32}^{(1)} = a_{32} - a_{31}a_{12}^{(1)} = 4 - 3 \times 0.5 = 2.5$$

$$j = 3;\ a_{33}^{(1)} = a_{33} - a_{31}a_{13}^{(1)} = 7 - 3 \times 1.5 = 2.5$$

$$b_3^{(1)} = b_3 - a_{31}b_1^{(1)} = 14 - 3 \times 3 = 5$$

$$a_{3j}^{(2)} = a_{3j}^{(1)} - a_{32}{}^{(1)}a_{2j}^{(2)} \qquad j = 3,\ n$$

$$j = 3;\ a_{33}^{(2)} = a_{33}^{(1)} - a_{32}^{(1)}\, a_{23}^{(2)} = 2.5 - 2.5 \times 0.5 = 1.25$$

$$b_3^{(2)} = b_3^{(1)} - a_{32}^{(1)}b_2^{(2)} = 5 - 2.5 \times 1.5 = 1.25$$

$$a_{3j}^{(3)} = \left(\frac{1}{a_{33}^{(2)}}\right)a_{3j}^{(2)} \qquad j = 4,\ n$$

$$b_3^{(3)} = \left(\frac{1}{a_{33}^{(2)}}\right)b_3^{(2)} = \left(\frac{1}{1.25}\right) \times 1.25 = 1$$

$$\begin{bmatrix} 1 & a_{12}^{(1)} & a_{13}^{(1)} & b_1^{(1)} \\ & 1 & a_{23}^{(2)} & b_2^{(2)} \\ & & 1 & b_3^{(3)} \end{bmatrix} = \begin{bmatrix} 1 & 0.5 & 1.5 & 3 \\ & 1 & 0.5 & 1.5 \\ & & 1 & 1 \end{bmatrix}$$

The solution can now be obtained by back substitution.

$$x_n = b_n^{(n)}$$

$$n = 3;$$

$$x_3 = b_3^{(3)} = 1$$

$$x_3 = 1$$

$$x_{n-1} = b_{n-1}^{(n-1)} - a_{n-1,n}^{(n-1)} \cdot x_n$$

$$x_{3-1} = b_{3-1}^{(3-1)} - a_{3-1,3}^{(3-1)} \cdot x_3$$

$$x_2 = b_2^{(2)} - a_{2,3}^{(2)} \cdot x_3 = 1.5 - 0.5 \times 1 = 1$$

$$x_2 = 1$$

$$x_i = b_i^{(i)} - \sum_{j=i+1}^{n} a_{ij}^{(i)} \cdot x_j$$

$$x_1 = b_1^{(1)} - [a_{12}^{(1)}x_2 + a_{13}^{(1)}x_3] = 3 - [0.5 \times 1 + 1.5 \times 1] = 1$$

$$x_1 = 1$$

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

1.8.4 Triangular Decomposition of Table of Factors

If the forward operations on b had been recorded so that they could be repeated, it is obvious that with this record and the upper triangle Eq. (1.65) for the back substitution, Eq. (1.60) could be solved for any vector b without repeating the triangularization. The recording of the forward operations, however, is trivial. Each forward operation is completely defined by the row and column coordinates and value of a single element $a_{ij}^{(j-1)}, i > j,$ that occurs in the process. Therefore, it is unnecessary to do anything to record these elements except to leave them.

The rules for recording the forward operations of triangularization are:

1. When a term $1/a_{ii}^{(i-1)}$ is computed, store it in the location ii.
2. Leave very derived term $a_{ij}^{(j-1)}, i > j,$ in the lower triangle Eq. (1.65).

Since the forward as well as the back substitution operations are recorded in this scheme, it is no longer necessary to include the vector b. The final result of triangularizing A and recording the forward operations is symbolized in Eq. (1.67).

$$\begin{matrix} d_{11} & u_{12} & u_{13} & \cdots & u_{1n} \\ l_{21} & d_{22} & u_{23} & \cdots & u_{2n} \\ l_{31} & l_{32} & d_{33} & \cdots & u_{3n} \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ l_{n1} & l_{n2} & l_{n3} & \cdots & d_{nn} \end{matrix} \tag{1.67}$$

The elements of Eq. (1.67) defined in terms of the derived systems of A in Eq. (1.61) to Eq. (1.65), are:

$$\begin{aligned} d_{ii} &= \frac{1}{a_{ii}^{(i-1)}} \\ u_{ij} &= a_{ij}^{(i)} \qquad i < j \\ l_{ij} &= a_{ij}^{(j-1)} \quad\; i > j \end{aligned} \tag{1.68}$$

The matrix brackets are omitted in Eq. (1.67) to emphasize that the array is not strictly a matrix in the same sense as the preceding examples, but only a scheme of recording. It will be referred to as the table of factors. In the literature this result is frequently shown as a factoring of the inverse matrix into the product of a lower and an upper triangular matrix, but it is more suitable for this discussion to consider it as a table of factors.

It is convenient in symbolizing the operations for obtaining direct solutions to define some special matrices in terms of the elements of the table of factors (1.67). The following nonsingular matrices differ from the unit matrix only in the row or column indicated.

$$D_i\text{: Row } i = (0,0,\ldots\ 0,\ d_{ii},\ 0,\ \ldots\ 0,\ 0)$$
$$L_i\text{: Col } i = (0,0,\ldots\ 0,1,\ -l_{i+1,i},\ -l_{i+2,i}\ \ldots\ -l_{n-1,i},\ -l_{n,i})^T$$
$$L_i\text{: Row } i = (-l_{i,1},\ -l_{i,2}\ \ldots\ -l_{i,i-1},\ 1,\ 0,\ \ldots\ 0,\ 0) \quad (1.69)$$
$$U_i\text{: Row } i = (0,0,\ \ldots\ 0,\ 1,\ -u_{i,i+1},\ -u_{i,i+2}\ \ldots\ -u_{i,n-1},\ -u_{i,n})$$
$$U_i\text{: Col } i = (-u_{1,i},\ -u_{2,i},\ \ldots\ -u_{i-1,i},\ 1,0,\ \ldots\ 0,0)^T$$

The inverses of these matrices are trivial. The inverse of the matrix D_i involves only the reciprocal of the element d_{ii}. The inverses of the matrices L_i, L_i^*, U_i and U_i* involve only a reversal of algebraic signs of the off-diagonal elements.

The forward and back substitution operations on the column vector b that transform it to x can be expressed as premultiplications by matrices D_i, or L L_i^* and U_i or U_i^*. Thus the solution of $AX = b$ can be expressed as indicated in Eq. (1.70a) to 1.70d).

$$U_1U_2\ \ldots\ U_{n-2}U_{n-1}D_nL_{n-1}D_{n-1}L_{n-2}\ \ldots\ L_2D_2L_1D_1b\ = A^{-1}\ b = x \quad (1.70a)$$
$$U_1U_2\ \ldots\ U_{n-2}U_{n-1}D_nL_n^*D_{n-1}L_{n-1}^*\ \ldots\ L_3^*D_2L_2^*D_1b\ = A^{-1}\ b = x \quad (1.70b)$$
$$U_2^*U_3^*\ \ldots\ U_{n-1}^*U_n^*D_nL_{n-1}D_{n-1}L_{n-2}\ \ldots\ L_2D_2L_1D_1b\ = A^{-1}\ b = x \quad (1.70c)$$
$$U_2^*U_3^*\ \ldots\ U_{n-1}^*U_n^*D_nL_n^*D_{n-1}L_{n-1}^*\ \ldots\ L_3^*D_2L_2^*D_1b\ = A^{-1}\ b = x \quad (1.70d)$$

EXAMPLE 1.13 Solve the following equations using triangular decomposition with table of factors.

$$2x_1 + x_2 + 3x_3 = 6$$
$$2x_1 + 3x_2 + 4x_3 = 9$$
$$3x_1 + 4x_2 + 7x_3 = 14$$

Solution:

In the matrix form $\begin{bmatrix} 2 & 1 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 7 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 6 \\ 9 \\ 14 \end{bmatrix}$

$$\therefore \qquad A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} 2 & 1 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 7 \end{bmatrix}$$

and augmented matrix $\begin{bmatrix} a_{11} & a_{12} & a_{13} & b_1 \\ a_{21} & a_{22} & a_{23} & b_2 \\ a_{31} & a_{32} & a_{33} & b_3 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 3 & 6 \\ 2 & 3 & 4 & 9 \\ 3 & 4 & 7 & 14 \end{bmatrix}$

Using the Gauss elimination method, we will get (from Example 1.12)

$$\begin{bmatrix} 1 & a_{12}^{(1)} & a_{13}^{(1)} & b_1^{(1)} \\ & 1 & a_{23}^{(2)} & b_2^{(2)} \\ & & 1 & b_3^{(3)} \end{bmatrix} = \begin{bmatrix} 1 & 0.5 & 1.5 & 3 \\ & 1 & 0.5 & 1.5 \\ & & 1 & 1 \end{bmatrix}$$

Table of factors

$$d_{ii} = \frac{1}{a_{ii}^{(i-1)}}$$

Therefore,

$$d_{11} = \frac{1}{a_{11}^{(0)}} = \left(\frac{1}{2}\right) = 0.5$$

$$d_{22} = \frac{1}{a_{22}^{(1)}} = \left(\frac{1}{2}\right) = 0.5$$

$\therefore\ a_{22}^{(1)}$ value obtained from Example 1.12

$$d_{33} = \frac{1}{a_{33}^{(2)}} = \left(\frac{1}{1.25}\right) = 0.8$$

$\therefore\ a_{33}^{(2)}$ value obtained from Example 1.12

$$u_{ij} = a_{ij}^{(i)} \qquad i < j$$

$$u_{12} = a_{12}^{(1)} = 0.5$$

$$u_{13} = a_{13}^{(1)} = 1.5$$

$$u_{23} = a_{23}^{(2)} = 0.5$$

$$l_{ij} = u_{ij}^{(j-1)} \qquad i > j$$

$$l_{21} = a_{21}^{(0)} = 2$$

$$l_{31} = a_{31}^{(0)} = 3$$

$$l_{32} = a_{32}^{(1)} = 2.5$$

$$\therefore\ a_{32}^{(1)} = a_{32} - a_{31}a_{12}^{(1)} = 4 - 3 \times 0.5 = 2.5$$

The table of factors for A is

$$\begin{matrix} d_{11} & u_{12} & u_{13} \\ l_{21} & d_{22} & u_{23} \\ l_{31} & l_{32} & d_{33} \end{matrix} = \begin{matrix} 0.5 & 0.5 & 1.5 \\ 2 & 0.5 & 0.5 \\ 3 & 2.5 & 0.0 \end{matrix}$$

With b given, the equation $AX = b$ can be solved for x using the equation below, which is based on Eq. (1.70a)

$$U_1U_2 \ldots U_{n-2}U_{n-1}D_nL_{n-1}D_{n-1}L_{n-2} \ldots L_2D_2L_1D_1b = A^{-1}\, b = x$$

For $n = 3$

$$U_1U_2D_3L_2D_2L_1D_1b = A^{-1}\, b = X$$

$$U_1 = \begin{bmatrix} 1 & -u_{12} & -u_{13} \\ & 1 & \\ & & 1 \end{bmatrix} = \begin{bmatrix} 1 & -0.5 & -0.5 \\ & 1 & \\ & & 1 \end{bmatrix}$$

$$U_2 = \begin{bmatrix} 1 & & \\ & 1 & -u_{23} \\ & & 1 \end{bmatrix} = \begin{bmatrix} 1 & & \\ & 1 & -0.5 \\ & & 1 \end{bmatrix}$$

$$D_3 = \begin{bmatrix} 1 & & \\ & 1 & \\ & & d_{33} \end{bmatrix} = \begin{bmatrix} 1 & & \\ & 1 & \\ & & 0.8 \end{bmatrix}$$

$$L_2 = \begin{bmatrix} 1 & & \\ & 1 & \\ & -l_{32} & 1 \end{bmatrix} = \begin{bmatrix} 1 & & \\ & 1 & \\ & -2.5 & 1 \end{bmatrix}$$

$$D_2 = \begin{bmatrix} 1 & & \\ & d_{22} & \\ & & 1 \end{bmatrix} = \begin{bmatrix} 1 & & \\ & 0.5 & \\ & & 1 \end{bmatrix}$$

$$L_1 = \begin{bmatrix} 1 & & \\ -l_{21} & 1 & \\ -l_{31} & & 1 \end{bmatrix} = \begin{bmatrix} 1 & & \\ -2 & 1 & \\ -3 & & 1 \end{bmatrix}$$

$$D_1 = \begin{bmatrix} d_{11} & & \\ & 1 & \\ & & 1 \end{bmatrix} = \begin{bmatrix} 0.5 & & \\ & 1 & \\ & & 1 \end{bmatrix}$$

$$b = \begin{bmatrix} 6 \\ 9 \\ 14 \end{bmatrix}$$

$$\underset{U_1}{\begin{bmatrix} 1 & -0.5 & -0.5 \\ & 1 & \\ & & 1 \end{bmatrix}} \underset{U_2}{\begin{bmatrix} 1 & & \\ & 1 & -0.5 \\ & & 1 \end{bmatrix}} \underset{D_3}{\begin{bmatrix} 1 & & \\ & 1 & \\ & & 0.8 \end{bmatrix}} \underset{L_2}{\begin{bmatrix} 1 & & \\ & 1 & \\ & -2.5 & 1 \end{bmatrix}} \underset{D_2}{\begin{bmatrix} 1 & & \\ & 0.5 & \\ & & 1 \end{bmatrix}}$$

$$\underset{L_1}{\begin{bmatrix} 1 & & \\ -2 & 1 & \\ -3 & & 1 \end{bmatrix}} \underset{D_1}{\begin{bmatrix} 0.5 & & \\ & 1 & \\ & & 1 \end{bmatrix}} \underset{b}{\begin{bmatrix} 6 \\ 9 \\ 14 \end{bmatrix}} = \underset{X}{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}}$$

EXAMPLE 1.14 Use the *LU* decomposition or triangular factorization method to solve the following simultaneous equations.

$$2x_1 + x_2 + 3x_3 = 6$$
$$2x_1 + 3x_2 + 4x_3 = 9$$
$$3x_1 + 4x_2 + 7x_3 = 14$$

Solution:

In the matrix form $\begin{bmatrix} 2 & 1 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 7 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 6 \\ 9 \\ 14 \end{bmatrix}$

$$A \quad = \quad L \quad U$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} l_{11} & & \\ l_{21} & l_{22} & \\ l_{31} & l_{32} & l_{33} \end{bmatrix}\begin{bmatrix} 1 & u_{12} & u_{13} \\ & 1 & u_{23} \\ & & 1 \end{bmatrix}$$

$$\begin{array}{lll} a_{11} = l_{11} & a_{12} = l_{11}u_{12} & a_{13} = l_{11}u_{13} \\ a_{21} = l_{21} & a_{22} = l_{21}u_{12} + l_{22} & a_{23} = l_{21}u_{13} + l_{22}u_{23} \\ a_{31} = l_{31} & a_{32} = l_{31}u_{12} + l_{32} & a_{33} = l_{31}u_{13} + l_{32}u_{23} + l_{33} \end{array}$$

$$\begin{bmatrix} 2 & 1 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 7 \end{bmatrix} = \begin{bmatrix} l_{11} & & \\ l_{21} & l_{22} & \\ l_{31} & l_{32} & l_{33} \end{bmatrix}\begin{bmatrix} 1 & u_{12} & u_{13} \\ & 1 & u_{23} \\ & & 1 \end{bmatrix}$$

The first row of the L & U matrices on the right side

$$a_{11} = l_{11} \Rightarrow l_{11} = a_{11} = 2$$

$$a_{12} = l_{11}u_{12} \Rightarrow u_{12} = \frac{a_{12}}{l_{11}} = \frac{1}{2} = 0.5$$

$$a_{13} = l_{11}u_{13} \Rightarrow u_{13} = \frac{a_{13}}{l_{11}} = \frac{3}{2} = 1.5$$

The second row of the L & U matrices on the right side

$$a_{21} = l_{21} \Rightarrow l_{21} = a_{21} = 2$$

$$a_{22} = l_{21}u_{12} + l_{22} \Rightarrow l_{22} = a_{22} - l_{21}u_{12} = 3 - 2 \times 0.5 = 2$$

$$a_{23} = l_{21}u_{13} + l_{22}u_{23} \Rightarrow u_{23} = \frac{a_{23} - l_{21}u_{13}}{l_{22}} = \frac{4 - 2 \times 1.5}{2} = 0.5$$

The third row of the L & U matrices on the right side

$$a_{31} = l_{31} \Rightarrow l_{31} = a_{31} = 3$$

$$a_{32} = l_{31}u_{12} + l_{32} \Rightarrow l_{32} = a_{32} - l_{31}u_{12} = 4 - 3 \times 0.5 = 2.5$$

$$a_{33} = l_{31}u_{13} + l_{32}u_{23} + l_{33} \Rightarrow l_{33} = a_{33} - l_{31}u_{13} - l_{32}u_{23}$$
$$= 7 - 3 \times 1.5 - 2.5 \times 0.5 = 1.25$$

$$L = \begin{bmatrix} l_{11} & & \\ l_{21} & l_{22} & \\ l_{31} & l_{32} & l_{33} \end{bmatrix} = \begin{bmatrix} 2 & & \\ 2 & 2 & \\ 3 & 2.5 & 1.25 \end{bmatrix}$$

$$U = \begin{bmatrix} 1 & u_{12} & u_{13} \\ & 1 & u_{23} \\ & & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0.5 & 1.5 \\ & 1 & 0.5 \\ & & 1 \end{bmatrix}$$

$$[A]\ [x] = [b]$$

And if $$[A] = [L]\ [U]$$

Then first solving $$[L]\ [z] = [b]$$

$$\begin{bmatrix} 2 & & \\ 2 & 2 & \\ 3 & 2.5 & 1.25 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 6 \\ 9 \\ 14 \end{bmatrix}$$

$$2z_1 = 6$$
$$z_1 = 3$$
$$2z_1 + 2z_2 = 9$$
$$z_2 = 1.5$$
$$3z_1 + 2.5z_2 + 1.25z_3 = 14$$
$$z_3 = 1$$

And then $$[U]\ [x] = [z]$$

$$\begin{bmatrix} 1 & 0.5 & 1.5 \\ & 1 & 0.5 \\ & & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 1.5 \\ 1 \end{bmatrix}$$

$$x_3 = 1$$
$$x_2 + 0.5x_3 = 1.5$$
$$x_2 = 1$$
$$x_1 + 0.5x_2 + 1.5x_3 = 3$$
$$x_1 = 1$$

$\therefore$ $$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

1.8.5 Bi-factorization Method

If A is a sparse matrix, the set of n linear equations, $AX = b$, can be solved effectively by using the bi-factorization method. The inverse of A can be expressed by a multiple product of $2n$ factor matrices.

$$A^{-1} = R^{(1)}R^{(2)} \ldots R^{(n)}L^{(n)} \ldots L^{(2)}L^{(1)} \tag{1.71}$$

In order to find L and R, the following sequence of intermediate matrices is introduced.

$$A^{(0)} = A$$
$$A^{(1)} = L^{(1)}A^{(0)}R^{(1)}$$
$$\vdots$$
$$A^{(j)} = L^{(j)}A^{(j-1)}R^{(j)}$$
$$\vdots$$
$$A^{(j)} = L^{(j)}A^{(j-1)}R(j)$$
$$\vdots$$
$$A^{(n)} = L^{(n)}A^{(n-1)}R^{(n)} = I$$

where the reduced matrix $A^{(j)}$ has elements defined by the following equations

$$a_{jj}^{(j)} = 1 \quad ; \quad a_{ij}^{(j)} = a_{ik}^{(j)} = 0$$

$$a_{ik}^{(j)} = a_{ik}^{(j-1)} - \frac{a_{ij}^{(j-1)} \cdot a_{jk}^{(j-1)}}{a_{jj}^{(j-1)}}$$

j being the pivotal index and $i, k = (j + 1), ..., n$.

The left-hand factor matrices $L^{(j)}$ are very sparse and differ from the unity matrix in column j only.

$$L^{(j)} = \begin{bmatrix} 1 & & & & \\ & 1 & & & \\ & & L_{ij}^{(j)} & & \\ & & \vdots & 1 & \\ & & L_{nj}^{(j)} & & 1 \end{bmatrix}$$

where

$$L_{ij}^{(j)} = \frac{1}{a_{jj}^{(j-1)}}$$

and

$$L_{ij}^{(j)} = -\frac{a_{ij}^{(j-1)}}{a_{jj}^{(j-1)}} \qquad i = (j + 1), ..., n$$

The right-hand factor matrices $R^{(j)}$ are also very sparse and differ from the unity matrix in row j only.

$$R^{(j)} = \begin{bmatrix} 1 & & & & \\ & 1 & & & \\ & R_{j,j+1}^{(j)} & \cdots & R_{jn}^{(j)} & \\ & & 1 & & \\ & & & 1 & \end{bmatrix}$$

where

$$R_{jk}^{(f)} = -\frac{a_{jk}^{(j-1)}}{a_{jj}^{(j-1)}} \qquad k = (j+1), \ldots, n$$

For a symmetric matrix A, the structures of left (L) and right (R) factor matrices are the same and can therefore be stored effectively in the matrix F as follows:

$$F = \begin{bmatrix} \ddots & & R \\ & \ddots & \\ L & & \ddots \end{bmatrix}$$

Solution to Eq. (1.71) can be written as

$$X = R^{(1)}R^{(2)} \ldots R^{(n)}L^{(n)} \ldots L^{(2)}L^{(1)}b \tag{1.72}$$

EXAMPLE 1.15 Solve the following simultaneous equations using the bi-factorization method.

$$2x_1 + x_2 + 3x_3 = 6$$
$$2x_1 + 3x_2 + 4x_3 = 9$$
$$3x_1 + 4x_2 + 7x_3 = 14$$

Solution: To find $X = R^{(1)}R^{(2)}L^{(3)}L^{(2)}L^{(1)}b$

In the matrix form $AX = b$

$$\begin{bmatrix} 2 & 1 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 7 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 6 \\ 9 \\ 14 \end{bmatrix}$$

$$A = A^{(0)} = \begin{bmatrix} a_{11}^{(0)} & a_{12}^{(0)} & a_{13}^{(0)} \\ a_{21}^{(0)} & a_{22}^{(0)} & a_{23}^{(0)} \\ a_{31}^{(0)} & a_{32}^{(0)} & a_{33}^{(0)} \end{bmatrix} = \begin{bmatrix} 2 & 1 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 7 \end{bmatrix}$$

Step 1:

$$L^{(1)}A^{(0)}R^{(1)} = A^{(1)}$$

$$\begin{bmatrix} L_{11}^{(1)} & . & . \\ L_{21}^{(1)} & 1 & . \\ L_{31}^{(1)} & . & 1 \end{bmatrix} \times A^{(0)} \times \begin{bmatrix} 1 & R_{12}^{(1)} & R_{13}^{(1)} \\ . & 1 & . \\ . & . & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & a_{22}^{(1)} & a_{23}^{(1)} \\ 0 & a_{32}^{(1)} & a_{33}^{(1)} \end{bmatrix}$$

$$L_{11}^{(1)} = \frac{1}{a_{11}^{(0)}}$$

$$L_{11}^{(1)} = \frac{1}{2} = 0.5$$

$$L_{i1}^{(1)} = -\frac{a_{i1}^{(0)}}{a_{11}^{(0)}} \qquad (i = 2, 3)$$

$$i = 2;\ L_{21}^{(1)} = -\frac{a_{21}^{(0)}}{a_{11}^{(0)}} = \frac{-2}{2} = -1$$

$$i = 3;\ L_{31}^{(1)} = -\frac{a_{31}^{(0)}}{a_{11}^{(0)}} = \frac{-3}{2} = -1.5$$

$$R_{1i}^{(1)} = -\frac{a_{1i}^{(0)}}{a_{11}^{(0)}} \qquad (i = 2,\ 3)$$

$$i = 2;\ R_{12}^{(1)} = -\frac{a_{12}^{(0)}}{a_{11}^{(0)}} = \frac{-1}{2} = -0.5$$

$$i = 3;\ R_{13}^{(1)} = -\frac{a_{13}^{(0)}}{a_{11}^{(0)}} = \frac{-3}{2} = -1.5$$

$$a_{ij}^{(1)} = a_{ij}^{(0)} - \frac{a_{i1}^{(0)} a_{1j}^{(0)}}{a_{11}^{(0)}} \qquad (i = 2,\ 3 \text{ and } j = 2,\ 3)$$

$$i = 2;\ j = 2;\ a_{22}^{(1)} = a_{22}^{(0)} - \frac{a_{21}^{(0)} a_{12}^{(0)}}{a_{11}^{(0)}} = 3 - \frac{2 \times 1}{2} = 2$$

$$i = 2;\ j = 3;\ a_{23}^{(1)} = a_{23}^{(0)} - \frac{a_{21}^{(0)} a_{13}^{(0)}}{a_{11}^{(0)}} = 4 - \frac{2 \times 3}{2} = 1$$

$$i = 3;\ j = 2;\ a_{32}^{(1)} = a_{32}^{(0)} - \frac{a_{31}^{(0)} a_{12}^{(0)}}{a_{11}^{(0)}} = 4 - \frac{3 \times 1}{2} = 2.5$$

$$i = 3;\ j = 3;\ a_{33}^{(1)} = a_{33}^{(0)} - \frac{a_{31}^{(0)} a_{13}^{(0)}}{a_{11}^{(0)}} = 7 - \frac{3 \times 3}{2} = 2.5$$

$$L^{(1)} A^{(0)} R^{(1)} = A^{(1)}$$

$$\begin{bmatrix} L_{11}^{(1)} & . & . \\ L_{21}^{(1)} & 1 & . \\ L_{31}^{(1)} & . & 1 \end{bmatrix} \times A^{(0)} \times \begin{bmatrix} 1 & R_{12}^{(1)} & R_{13}^{(1)} \\ . & 1 & . \\ . & . & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & a_{22}^{(1)} & a_{23}^{(1)} \\ 0 & a_{32}^{(1)} & a_{33}^{(1)} \end{bmatrix}$$

$$\begin{bmatrix} 0.5 & . & . \\ -1 & 1 & . \\ -1.5 & . & 1 \end{bmatrix} \times \begin{bmatrix} 2 & 1 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 7 \end{bmatrix} \times \begin{bmatrix} 1 & -0.5 & -1.5 \\ . & 1 & . \\ . & . & 1 \end{bmatrix} = \begin{bmatrix} 1 & . & . \\ . & 2 & 1 \\ . & 2.5 & 2.5 \end{bmatrix}$$

Step 2:

$$L^{(2)}A^{(1)}R^{(2)} = A^{(2)}$$

$$\begin{bmatrix} 1 & . & . \\ . & L_{22}^{(2)} & . \\ . & L_{32}^{(2)} & 1 \end{bmatrix} \times A^{(1)} \times \begin{bmatrix} 1 & . & . \\ . & 1 & R_{23}^{(1)} \\ . & . & 1 \end{bmatrix} = \begin{bmatrix} 1 & . & . \\ . & 1 & . \\ . & . & a_{33}^{(1)} \end{bmatrix}$$

$$L_{22}^{(2)} = \frac{1}{a_{22}^{(1)}}$$

$$L_{12}^{(2)} = \frac{1}{2} = 0.5$$

$$L_{i2}^{(2)} = -\frac{a_{i2}^{(1)}}{a_{22}^{(1)}} \qquad (i = 3,\ 4)$$

$$i = 3;\ L_{32}^{(2)} = -\frac{a_{32}^{(1)}}{a_{22}^{(1)}} = \frac{-2.5}{2} = -1.25$$

$$R_{2i}^{(2)} = -\frac{a_{2i}^{(0)}}{a_{22}} \qquad (i = 3,\ 4)$$

$$i = 3;\ R_{23}^{(2)} = -\frac{a_{23}^{(1)}}{a_{22}^{(1)}} = \frac{-1}{2} = -0.5$$

$$a_{ij}^{(2)} = a_{ij}^{(2)} - \frac{a_{i2}^{(1)}a_{2j}^{(1)}}{a_{22}^{(1)}} \qquad (i = 3 \text{ and } j = 3)$$

$$i = 3;\ j = 3;\ a_{33}^{(2)} = a_{33}^{(1)} - \frac{a_{32}^{(1)}a_{23}^{(1)}}{a_{22}^{(1)}} = 2.5 - \frac{2.5 \times 1}{2} = 1.25$$

$$L^{(2)}A^{(1)}R^{(2)} = A^{(2)}$$

$$\begin{bmatrix} 1 & . & . \\ . & L_{22}^{(2)} & . \\ . & L_{32}^{(2)} & 1 \end{bmatrix} \times A^{(1)} \times \begin{bmatrix} 1 & . & . \\ . & 1 & R_{23}^{(2)} \\ . & . & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & . \\ 0 & . & a_{33}^{(2)} \end{bmatrix}$$

$$\begin{bmatrix} 1 & . & . \\ . & 0.5 & . \\ . & -1.25 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & . & . \\ . & 2 & 1 \\ . & 2.5 & 2.5 \end{bmatrix} \times \begin{bmatrix} 1 & . & . \\ . & 1 & -0.5 \\ . & . & 1 \end{bmatrix} = \begin{bmatrix} 1 & . & . \\ . & 1 & . \\ . & . & 1.25 \end{bmatrix}$$

Step 3:

$$L^{(3)}A^{(2)}R^{(3)} = A^{(3)}$$

$$\begin{bmatrix} 1 & . & . \\ . & 1 & . \\ . & . & L_{33}^{(3)} \end{bmatrix}$$

$$L_{33}^{(3)} = \frac{1}{a_{33}^{(2)}} = \frac{1}{1.25} = 0.8$$

$$L^{(3)} = \begin{bmatrix} 1 & . & . \\ . & 1 & . \\ . & . & 0.8 \end{bmatrix}$$

$$X = \quad R^{(1)} \quad R^{(2)} \quad L^{(3)} \quad L^{(2)} \quad L^{(1)} \quad b$$

$$X = \begin{bmatrix} 1 & -0.5 & -1.5 \\ . & 1 & . \\ . & . & 1 \end{bmatrix} \begin{bmatrix} 1 & . & . \\ . & 1 & -0.5 \\ . & . & 1 \end{bmatrix} \begin{bmatrix} 1 & . & . \\ . & 1 & . \\ . & . & 0.8 \end{bmatrix} \begin{bmatrix} 1 & . & . \\ . & 0.5 & . \\ . & -1.25 & 1 \end{bmatrix} \begin{bmatrix} 0.5 & . & . \\ -1 & 1 & . \\ -1.5 & . & 1 \end{bmatrix} \begin{bmatrix} 6 \\ 9 \\ 14 \end{bmatrix}$$

$$X = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Comparison of methods for triangularization and bi-factorization

- These are identical methods based on removing Gauss.
- The number of arithmetic operations is equal in both methods.
- Since the factors raised by both the methods are different, the method of bi-factorization requires less memory, less indexing and simple arithmetic operations.
- In the triangular decomposition *LU*s are not symmetric. *LDU*s in the decomposition are symmetrical, but require a larger number of operations.
- The product of triangular matrices is also the matrix. The product of matrices is the matrix bifactors inverse, which facilitates the attainment of the solution.
- The bi-factorization is of particular interest for those sparse matrices, which have predominantly main diagonal elements, or arrays that are not symmetrical, but have a structure symmetric sparsity.

1.8.6 Sparsity and Optimal Ordering Scheme

When the matrix to be triangularized is sparse, the order in which rows are processed affects the number of nonzero terms in the resultant upper triangle.

If a programming scheme is used which processes and stores only the nonzero terms, a great savings in operations and computer memory can be achieved by keeping the table of factors as sparse as possible. The absolute optimal order of elimination would result in the least possible terms in the table of factors. An efficient algorithm for determining the absolute optimal order has not been developed, and it appears to be a practical impossibility. However, several effective schemes have been developed for determining the near-optimal orders.

Schemes for near-optimal ordering

The inspection algorithms for near-optimal ordering are applicable to sparse matrices that are symmetric in pattern of nonzero off-diagonal terms, i.e. if a_{ij} is nonzero, then a_{ji} also is nonzero but not necessarily equal to a_{ij}. These are the matrices that occur most frequently in network problems. From the standpoint of programming efficiency, the algorithms should be applied before, rather than during, the triangularization. It is assumed in what follows that the matrix rows are originally numbered according to some external criterion and then renumbered according to the inspection algorithm. Eliminations are then performed in ascending sequence of the renumbered system.

The descriptions of three schemes for renumbering in near-optimal order are the following. They are listed in increasing order of programming complexity, execution time, and optimality.

***Scheme* 1:** Number the rows of the coefficient matrix A according to the number of nonzero off-diagonal terms before elimination. In this scheme the rows with only one off-diagonal term are numbered first, those with two off-diagonal terms are numbered second, etc. and those with the most off-diagonal terms are numbered last.

From the network point of view, the nodes are numbered, starting with that having the fewest connected branches (i.e. minimum degree). This method does not take into account anything that happens during the elimination process but it is simple to program and fast to execute. The only information needed here is a list of the number of nonzero terms in each row of the original matrix.

***Scheme* 2:** Number the rows of the coefficient matrix A so that at each step of the process the next row to be operated upon is the one with the fewest nonzero terms. If more than one row meets this criterion, select anyone.

From the network point of view, the nodes are numbered, so that at each step of the elimination the next node to be eliminated is the one having the fewest connected branches (i.e. minimum degree). This method requires a simulation of elimination process to take into account the changes in the node branch connection effected at each. This scheme requires a simulation of the effects on the accumulation of nonzero terms of the elimination process. Input information is a list by rows of the column numbers of the nonzero off-diagonal terms, i.e. branches. This scheme, though takes longer time, is definitely better.

***Scheme* 3:** Number the rows so that at each step of the elimination process the next row to be operated upon is the one that will introduce the fewest new nonzero terms. If more than one row meets this criterion, select anyone.

From the network point of view, the nodes are numbered, such that at each step of the elimination process the next node to be eliminated is the one that will introduce the fewest row equivalent of every feasible alternative, i.e. new links at each step. Input information is the same as that of scheme (2).

Advantages of the above schemes: The comparative advantages of these schemes are influenced by the network topology and size and the number of direct solutions required. The only virtue of scheme (1) is its simplicity and speed. For nodal equations of power networks, scheme (2) is enough and better than scheme (1), to justify the additional time required for its execution. Scheme (3) does not appear to be enough, and not better than scheme (2) to justify its use for power networks, but it is known to be effective for other networks.

Comparative advantages for a sparse matrix

When A is a sparse matrix, the advantages of the factored form in addition to those previously listed are:

1. The table of factors can be obtained in a small fraction.
2. The storage requirement is small, permitting much larger systems to be solved.
3. Direct solutions can be obtained much faster unless the independent vector is extremely sparse.
4. Round-off error is reduced.
5. Modifications due to changes in the matrix can be made much faster.

The only **disadvantage** of the method is that it requires much more sophisticated programming techniques.

Review Questions

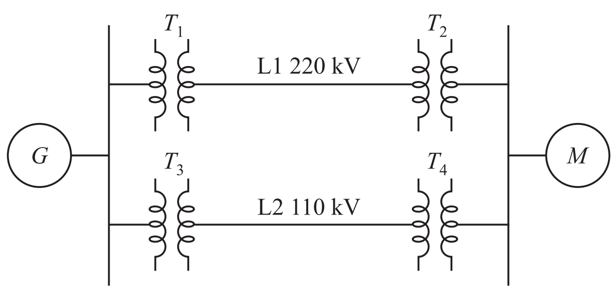
Part-A

1. What is power system?
2. What are the objectives of power system analysis?
3. What are the components of power system?
4. What is the modern power system?
5. What is complex power?
6. What is a bus?
7. Define per phase analysis.
8. Draw the per phase basis or modelling or representation of all components of power system.
9. What is an infinite bus bar?

10. What is single line diagram?
11. What is the purpose of using single line diagram?
12. What is impedance diagram? What are the approximations made in impedance diagram?
13. What is reactance diagram? What are the approximations made in reactance diagram?
14. Define per unit value.
15. What are the advantages of per unit system?
16. What is the need for base values?
17. Write the equation for per unit impedance if a change of base occurs.
18. A generator rated at 30 MVA, 11 kV has a reactance of 20%. Calculate its per unit reactance for a base of 50 MVA and 10 kV.
19. What is the new per unit impedance if the new base MVA is twice the old base MVA?
20. What is a primitive network?
21. What is a bus admittance matrix?
22. What are the methods available for forming the bus admittance matrix?
23. What is sparse matrix?
24. What are the advantages and disadvantages of sparse matrix?
25. Compare the methods of triangularization and bifactorization.

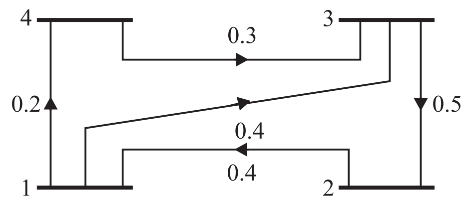
Part-B

1. The single line diagram of a three-phase power system is shown in the figure below. Draw its per unit impedance diagram.



G: 100 MVA, 33 kV, X = 20%
T_1: 50 MVA, 33/220 kV, X = 10%
T_2: 40 MVA, 220/33 kV, X = 5%
T_3: 30 MVA, 33/110 kV, X = 5.2%
T_4: 40 MVA, 110/33 kV, X = 5%
M: 80 MVA, 10.45 kV, X = 20%
Line 1 = 115 Ω; Line 2 = 40 Ω
Motor: 60 MVA, 33 kV, X = 20%

2. Form the matrix $[Y_{bus}]$ and compute the answer using the inspection method and singular transformation method. All the impedance values are in per unit.



3. For a power system network with the following data, compute the bus admittance matrix.

System data

Line	*Start*	*End*	*X-value*
G_1	1	0	1
G_2	5	0	1.25
L_1	1	2	0.4
L_2	1	3	0.5
L_3	2	3	0.25
L_4	2	5	0.2
L_5	3	4	0.125
L_6	4	5	0.2

4. For the power system network with the following data, compute the bus incident matrix and form the bus admittance matrix by the singular transformation method.

Bus code	*Per unit line impedance*	*Half line charging admittance in per unit*
1–2	$0.05 + j0.12$	$j0.025$
2–3	$0.06 + j0.4$	---
3–4	$0.75 + j0.25$	$j0.02$
1–3	$0.045 + j0.45$	$j0.015$
1–4	$0.015 + j0.05$	---

5. Find the L and U triangular factors of the symmetric matrix.

$$M = \begin{bmatrix} 2 & 1 & 3 \\ 1 & 5 & 4 \\ 3 & 4 & 7 \end{bmatrix}$$

6. Solve the following equations using the Gauss elimination method and verify the result using the bi-factorization method:

$$2x_1 + x_2 + 3x_3 = 5$$
$$1x_1 + 5x_2 + 4x_3 = 3$$
$$3x_1 + 4x_2 + 7x_3 = 12$$